

Diffraction Managed Solitons with Zero Mean Diffraction

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The problem of existence of soliton solutions in a diffraction managed NLS is considered in the case of zero mean diffraction. Existence of localized breather solutions for the averaged equation is shown using Ekeland's variational principle in the corresponding minimization of the Hamiltonian procedure. The corresponding minimizer is shown to be spatially well-localized and orbitally stable.

KEY WORDS: Diffraction managed nonlinear Schrödinger equation; localization; ground states.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Over the last several decades optical solitons have become a promising candidate for optical communication networks. Solitons are localized coherent structures, resulting from a balance of dispersive and nonlinear effects. Ever since 1960 when they were discovered and studied in the works of Zabusky and Kruskal, Zakharov and Shabat, Lax, Miura *et al.* and many others, optical spatial solitons have become one of the most exciting research areas in optics and nonlinear science. The equation, used to model the electromagnetic wave propagation in an optical waveguide is the nonlinear Schrödinger equation. In the last two decades, a technique that uses fibers with alternating sections having opposite dispersion was

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introduced. This technology, called dispersion management, proved to be incredibly successful in producing stable, soliton-like pulses. The idea is to use rapidly varying dispersion with approximately zero mean and small nonlinearity in hope that the balance between the small residual dispersion and the small nonlinearity will produce a soliton-like solutions. The equation that describes this problem is a variable-coefficient NLS equation usaly called dispersion managed NLS (DMNLS). There have been an enormous amount of technological advances in this direction and now there is a fully established theoretical understanding of the strong stability properties of the dispersion managed (DM) systems, see [8,9,11–15,19,23]. At the same time, discrete systems have found many applications in nonlinear optics. In discrete networks such as coupled waveguide arrays a balance of diffraction term that comes from the weak coupling between neighboring waveguides and the nonlinearity can produce discrete solitons. As in the case of continuous NLS and dispersion management, the technological motivation to study systems with varying diffraction is to achieve low power coherent structures. This emerging technology, called diffraction management, uses the geometry of the waveguide arrays to achieve periodic altering of the diffraction coefficient with a desired average diffraction. This technology was studied experimentally, see [4–6]. As in the continuous media, localized stable nonlinear waves are also found in the discrete case. They were first demonstrated numerically and then subsequently analyzed using the discrete nonlinear Schrödinger equation [2,16,17]. The model for a one-dimensional diffraction managed waveguide array is given by (see for example [16])

$$i \frac{\partial u_n}{\partial \xi} + \left(\frac{\tilde{d}(\xi/\varepsilon)}{\varepsilon} + \bar{d} \right) k(\Delta_d u)_n + (|u|^2 u)_n = 0, \quad (1)$$

where the total diffraction is given as a sum of a large rapidly varying component $\tilde{d}(\xi/\varepsilon)/\varepsilon$ and a small average component $\bar{d}$, and ε is a small positive parameter. As usual ξ is the distance down the waveguide, n -the location of the array element, $k = 1/2h^2$, $h > 0$ is the lattice spacing and Δ_d is the discrete Laplacian,

$$(\Delta_d u)_n = \sum_{j=1}^d (u_{n+e_j} + u_{n-e_j} - 2u_n).$$

Following the well established procedure for the continuous case [23], an averaged equation is constructed, for which stable ground state solutions are proven to exist using minimization of an associated Hamiltonian. The

Hamiltonian corresponding to this averaged equation is given by

$$\langle H \rangle = \varepsilon \left(\bar{d}k(-\Delta_d v, v) - \frac{1}{2} \int_0^1 \sum_{n=-\infty}^{\infty} |(e^{iz\Delta_d} v)_n|^4 dz \right).$$

In [16,17] the problem of constructing constrained minimizers of this Hamiltonian is studied in the case of nonnegative average dispersion and the existence of breather solitons is shown using a discrete version of Lion's concentration compactness principle. In this paper we concentrate on the case when the average dispersion has zero mean and we provide a shorter proof of the existence of constrained minimizers of the Hamiltonian. Another advantage of our method is that it allows us to show that the minimizing sequence is spatially well-localized and this is the main contribution of this paper.

Similarly to [19], in the case of zero mean dispersion we have to study the following minimization problem. Find a minimizer v with a fixed l^2 -norm λ for the functional $\varphi(v)$.

$$\begin{aligned} \varphi(v) &= - \int_0^1 \sum_{n=-\infty}^{\infty} |(e^{iz\Delta_d} v)_n|^4 dz \rightarrow \min \\ \text{subject to } \|v\|_{l^2} &= \lambda. \end{aligned} \quad (2)$$

We prove the following theorem.

Theorem 1. *For every $\lambda > 0$, the minimization problem (2) has a solution v . The minimizer v is spatially well-localized in the sense that for every N , there exists a constant C_N , so that $|v_m| \leq C_N(1 + |m|)^{-N}$. Moreover, it is orbitally stable.*

2. PRELIMINARIES

We use the sequence Banach spaces $l^p(\mathbf{Z}^d)$, $1 \leq p \leq \infty$, equipped with the norm

$$\|v\|_{l^p} = (\sum_{n \in \mathbf{Z}^d} |v_n|^p)^{1/p}. \text{ Note that } \|\cdot\|_{l^q} \leq \|\cdot\|_{l^p} \text{ for } 1 \leq p \leq q \leq \infty.$$

2.1. Representation of Δ_d and $e^{it\Delta_d}$ on $l^2(\mathbf{Z}^d)$

Following [16,20], we derive a formula for the action of the generator Δ_d and the semigroup $e^{it\Delta_d}$ on $l^2(\mathbf{Z}^d)$. We define the shift operator in the j th coordinate $S_j: l^2(\mathbf{Z}^d) \rightarrow l^2(\mathbf{Z}^d)$ (and its adjoint S_j^*) by

$$\begin{aligned} (S_j v)(n_1, \dots, n_d) &= v(n_1, \dots, n_j - 1, \dots, n_d), \\ (S_j^* v)(n_1, \dots, n_d) &= v(n_1, \dots, n_j + 1, \dots, n_d). \end{aligned}$$

In this setting the discrete Laplacian can be written using the shift operators as

$$\Delta_d(v) = \sum_{j=1}^d (S_j + S_j^* - 2Id)(v)$$

for every sequence $v \in l^2(\mathbf{Z}^d)$. We now recall some basic facts regarding the (discrete) Fourier transform. Associate to the sequence $v = (v_n)_{n \in \mathbf{Z}^d} \in l^2(\mathbf{Z}^d)$ its discrete Fourier transform

$$f(x) = \sum_{n \in \mathbf{Z}^d} v_n e^{2\pi i \langle n, x \rangle} \in L^2([0, 1]^d).$$

One can of course recover the coefficients v_n from the function f via the inverse discrete Fourier transform

$$v_n = \int_{[0, 1]^d} f(x) e^{-2\pi i \langle n, x \rangle} dx.$$

The Parseval formula simply states that the mapping $v \rightarrow f$ is an isometry between $l^2(\mathbf{Z}^d)$ and $L^2([0, 1]^d)$.

Using the one-to-one correspondence between v and f , we deduce the following formulas for the shift operator

$$\begin{aligned} (S_j f)(x) &= e^{2\pi i x_j} f(x), \\ (S_j^* f)(x) &= e^{-2\pi i x_j} f(x). \end{aligned}$$

It follows that (on the level of the Fourier transforms $f(x)$)

$$\begin{aligned} \Delta_d f(x) &= \sum_{j=1}^d (e^{2\pi i x_j} + e^{-2\pi i x_j} - 2) f(x) = -4 \sum_{j=1}^d \sin^2(\pi x_j) f(x), \\ e^{it\Delta_d} f(x) &= e^{-4it \sum_{j=1}^d \sin^2(\pi x_j)} f(x) \end{aligned}$$

In the sequence setup, one derives

$$(e^{it\Delta_d} v)_n = \sum_{m \in \mathbf{Z}^d} v_m \int_{[0, 1]^d} e^{-4it \sum_{j=1}^d \sin^2(\pi x_j)} e^{2\pi i \langle m-n, x \rangle} dx. \quad (3)$$

Using the fact that $\mathcal{F}: L^2([0, 1]^d) \rightarrow l^2(\mathbf{Z}^d)$ is an isometry together with its inverse and taking $v = \mathcal{F}^{-1} f$ yields

$$\|e^{it\Delta_d} v\|_{l^2(\mathbf{Z}^d)} = \|e^{it\Delta_d} f\|_{L^2([0, 1]^d)} = \|f\|_{L^2([0, 1]^d)} = \|v\|_{l^2(\mathbf{Z}^d)}, \quad (4)$$

that is $e^{it\Delta_d}$ is an isometry on $l^2(\mathbf{Z}^d)$ as well.

The following lemma is elementary, but it will be very useful.

Lemma 1. *For every integer N , there exists a constant $C_{N,d}$, so that for every bounded sequence v one has*

$$|(e^{it\Delta_d}v)_n| \leq C_{N,d} \max(1, t)^N \sum_{m \in \mathbb{Z}^d} \frac{|v_m|}{(1 + |m - n|)^N}. \quad (5)$$

In particular, every bounded sequence is mapped in a bounded one via $e^{it\Delta_d}$.

Proof. Taking absolute values in (3) allows one to reduce the proof to

$$\left| \int_{[0,1]^d} e^{-4it \sum_{j=1}^d \sin^2(\pi x_j)} e^{2\pi i \langle m-n, x \rangle} dx \right| \leq C_{N,d} \max(1, t)^N (1 + |m - n|)^{-N}.$$

If $m = n$, the estimate is obvious by taking absolute values inside the integral, while for $n \neq m$, we integrate by parts (note that the boundary terms disappear by the one periodicity of the integrand), gaining as many factors of $|m - n|^{-1}$ as we wish (and paying for each integration by parts with factors of magnitude less than $C_d t$). $\square$

2.2. Littlewood–Paley Projections in the Discrete Setting

Introduce the linear operator $P_l: l^2(\mathbb{Z}^d) \rightarrow l^2(\mathbb{Z}^d)$ by

$$P_l(v) = \sum_{|n|=2^{l-1}}^{2^{l+1}-1} v_n e_n$$

that is P_l is a projection onto $\text{span}\{e_n, 2^{l-1} \leq |n| < 2^{l+1}\}$.

Similarly, $P_{l,l+j}(v) = \sum_{|n|=2^{l-1}}^{2^{l+j+1}-1} v_n e_n$ and $P_{>l} = P_{l,\infty}$. We also need the following technical lemma.

Lemma 2. *For every integer $N > d + 1$, there exists a constant $C = C(N, d)$, so that for every sequence $z \in l^2(\mathbb{Z}^d)$ and for every $0 < t < 1$,*

$$\|P_l(e^{it\Delta_d}z)\|_{l^2} \leq C \left(2^{-lN} \|z\|_{l^2} + \|P_{l-2,l+2}z\|_{l^2} \right). \quad (6)$$

Proof. From (3) and Lemma 1

$$|(e^{it\Delta_d}z)_n| \leq C_N \sum_m |z_m| (1 + |m - n|)^{-N}.$$

Hence,

$$\begin{aligned}
 \|P_l(e^{it\Delta_d}z)\|_{l^2}^2 &\leq C_N \sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \left(\sum_m |z_m| (1 + |m - n|)^{-N} \right)^2 \\
 &\leq C_N \sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \left(\sum_{2^{l-2} \leq |m| \leq 2^{l+2}} |z_m| (1 + |m - n|)^{-N} \right)^2 \\
 &\quad + C_N \sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \left(\sum_{|m| \notin [2^{l-2}, 2^{l+2}]} |z_m| (1 + |m - n|)^{-N} \right)^2.
 \end{aligned}$$

For the first term on the right, by Cauchy–Schwartz and for $N > d + 1$,

$$\begin{aligned}
 &\sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \left(\sum_{2^{l-2} \leq |m| \leq 2^{l+2}} |z_m| (1 + |m - n|)^{-N} \right)^2 \\
 &\leq \sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \sum_{2^{l-2} \leq |m| \leq 2^{l+2}} |z_m|^2 (1 + |m - n|)^{-N} \\
 &\quad \times \sum_{2^{l-2} \leq |m| \leq 2^{l+2}} (1 + |m - n|)^{-N} \\
 &\leq C_{N,d} \sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \sum_{2^{l-2} \leq |m| \leq 2^{l+2}} \frac{|z_m|^2}{(1 + |m - n|)^N} \\
 &\leq C_{N,d} \sum_{2^{l-2} \leq |m| \leq 2^{l+2}} |z_m|^2,
 \end{aligned}$$

while for the second term,

$$\begin{aligned}
 &\sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \left(\sum_{|m| \notin [2^{l-2}, 2^{l+2}]} |z_m| (1 + |m - n|)^{-N} \right)^2 \\
 &\leq \sum_{2^{l-1} \leq |n| \leq 2^{l+1}} \sum_{|m| \notin [2^{l-2}, 2^{l+2}]} |z_m|^2 \sum_{|m| \notin [2^{l-2}, 2^{l+2}]} (1 + |m - n|)^{-2N} \\
 &\leq C_{N,d} 2^{ld} \sum_{|k| \geq 2^l} (1 + |k|^{-2N}) \|z\|_{l^2}^2 \leq C_{N,d} 2^{ld} 2^{-l(2N-d)} \|z\|_{l^2}^2 \\
 &\leq C_{N,d} 2^{2l(d-N)} \|z\|_{l^2}^2.
 \end{aligned}$$

The lemma is proved, if one selects sufficiently large N . □

As a corollary of Lemma 2, we need

Corollary 1. *Let $z_1, z_2, z_3 \in l^2(\mathbf{Z}^d)$. Then*

$$\begin{aligned} \left\| P_l[(e^{it\Delta_d} z_1)(e^{it\Delta_d} z_2)(e^{it\Delta_d} z_3)] \right\|_{l^2} &\leq C(2^{-lN} \max_{1 \leq j \leq 3} \|z_j\|_{l^2}^3 \\ &\quad + \max_{1 \leq j \leq 3} \|P_{l-2,l+2} z_j\|_{l^2}^3) \end{aligned}$$

Proof. We have

$$\begin{aligned} &\left\| P_l[(e^{it\Delta_d} z_1)(e^{it\Delta_d} z_2)(e^{it\Delta_d} z_3)] \right\|_{l^2} \\ &\leq \left(\sum_{2^{l-1} \leq |n| \leq 2^{l+1}} |(e^{it\Delta_d} z_1)_n|^2 |(e^{it\Delta_d} z_2)_n|^2 |(e^{it\Delta_d} z_3)_n|^2 \right)^{1/2} \\ &\leq C \|P_l e^{it\Delta_d} z_1\|_{l^6} \|P_l e^{it\Delta_d} z_2\|_{l^6} \|P_l e^{it\Delta_d} z_3\|_{l^6} \\ &\leq C \prod_{j=1}^3 \|P_l e^{it\Delta_d} z_j\|_{l^2}. \end{aligned}$$

According to Lemma 2, one gets

$$\left\| P_l[(e^{it\Delta_d} z_1)(e^{it\Delta_d} z_2)(e^{it\Delta_d} z_3)] \right\|_{l^2} \leq C_N \prod_{j=1}^3 (2^{-lN} \|z_j\|_{l^2} + \|P_{l-2,l+2} z_j\|_{l^2}).$$

Expand the product. Estimate all terms containing 2^{-lN} by $C_{N,d} 2^{-lN} \max_{1 \leq j \leq 3} \|z_j\|_{l^2}^3$. The remaining term is clearly smaller than $\max_{1 \leq j \leq 3} \|P_{l-2,l+2} z_j\|_{l^2}^3$. $\square$

Clearly, Corollary 1 may be stated and proved with any finite number of terms $z_1, \dots, z_m$, but that will not be necessary in our argument.

The next lemma states precise conditions for precompactness of a sequence in l^p .

Lemma 3. *The sequence $\{u^m\}$ is precompact in $l^p(\mathbf{Z}^d)$, $1 < p < \infty$ if and only if*

- $\sup_m \|u^m\|_{l^p} \leq C < \infty$
- $\lim_{N \rightarrow \infty} \limsup_{m \rightarrow \infty} \left(\sum_{|j| > N} |u_j^m|^p \right)^{1/p} = 0$.

Proof. The proof of this result is standard, see for example Theorem 3.1 [22]. $\square$

3. PROOF OF THEOREM 1

To prove the theorem we first need to show that $\inf_{\|v\|_{l^2}=\lambda} \varphi(v) > -\infty$. This is in fact easy to check, since

$$\sum_n \int_0^1 |(e^{it\Delta_d} v)_n|^4 dt \leq \sup_{t \in (0,1)} \|e^{it\Delta_d} v\|_{l^4}^4 \leq \sup_{t \in (0,1)} \|e^{it\Delta_d} v\|_{l^2}^4 = \|v\|_{l^2}^4 = \lambda^4,$$

which implies $\varphi(v) \geq -\lambda^4$

By Ekeland variational principle (see [12, 14, 19] for a similar argument in the continuous case), there exists a constant $c \neq 0$ (a “Lagrange multiplier”) and a sequence $u^m : \|u^m\|_{l^2} = \lambda$, so that

$$\lim_m \|cu^m + \varphi'(u^m)\|_{l^2} = 0. \quad (7)$$

Here $\varphi'(u^m)$ is the Gateaux derivative of the nonlinear functional φ evaluated at u^m . The precise formula for $\varphi'(v)$ is easy to compute (see [16] for the discrete case and [14, 19] for the continuous case). Namely,

$$\varphi'(v) = - \int_0^1 e^{-it\Delta_d} (|e^{it\Delta_d} v|^2 e^{it\Delta_d} v) dt. \quad (8)$$

$\{u^m\}$ on the other hand is what is referred to as a “minimizing sequence” for the problem. Next, we need a lemma.

Lemma 4. *Let $u, v \in l^2(\mathbf{Z}^d)$. There exists a constant C , so that*

$$|\varphi(u) - \varphi(v)| \leq C \|u - v\|_{l^2} (\|u\|_{l^2}^3 + \|v\|_{l^2}^3).$$

Proof. Start with

$$\begin{aligned} |\varphi(u) - \varphi(v)| &\leq \int_0^1 \sum_{n=-\infty}^{\infty} (|e^{it\Delta_d} u_n|^4 - |e^{it\Delta_d} v_n|^4) dt \\ &\leq \int_0^1 \sum_{n=-\infty}^{\infty} (|e^{it\Delta_d} u_n| - |e^{it\Delta_d} v_n|) (|e^{it\Delta_d} u_n| + |e^{it\Delta_d} v_n|) dt \\ &\quad + |e^{it\Delta_d} v_n| (|e^{it\Delta_d} u_n|^2 + |e^{it\Delta_d} v_n|^2) dt \\ &\leq \sup_t (\|e^{it\Delta_d} u\|_{l^4}^2 + \|e^{it\Delta_d} v\|_{l^4}^2) (\|e^{it\Delta_d} u\|_{l^4} + \|e^{it\Delta_d} v\|_{l^4}) \\ &\quad + \|e^{it\Delta_d} v\|_{l^4} \|e^{it\Delta_d} (u - v)\|_{l^4}. \end{aligned}$$

Now use that $\|e^{it\Delta_d}u\|_{l^4} \leq \|e^{it\Delta_d}u\|_{l^2} \leq \|u\|_{l^2}$ to get that

$$|\varphi(u) - \varphi(v)| \leq C\|u - v\|_{l^2}(\|u\|_{l^2}^3 + \|v\|_{l^2}^3).$$

□

Based on Lemma 4, it is easy to see that if we manage to show that for a subsequence $\{u^{m_k}\}$ $\|u^{m_k} - u\|_{l^2} \rightarrow 0$, then $\|u\|_{l^2} = \lim_k \|u^{m_k}\|_{l^2} = \lambda$ and

$$|\varphi(u^{m_k}) - \varphi(u)| \leq C\|u^{m_k} - u\|_{l^2}(\|u^{m_k}\|_{l^2}^3 + \|u\|_{l^2}^3) \leq C\lambda^3\|u^{m_k} - u\|_{l^2}.$$

implying that $\varphi(u) = \inf_{\|v\|_{l^2}=\lambda} \varphi(v)$, that is u solves the problem.

3.1. The Minimizing Sequence u^m has a Strongly Convergent Subsequence

In this section we will show that indeed $\|u^{m_k} - u\|_{l^2} \rightarrow 0$ for some subsequence $\{u^{m_k}\}_k$ of $\{u^m\}$. Our approach will be similar to [19], namely it will make use of the dispersive estimates directly, rather than using the concentration compactness techniques employed by Moeser [16] (for the case of positive average dispersion) and by Kunze [12] in the continuous case.

First, since the sequence $\{u^m\}$ is uniformly bounded, pick a weakly convergent subsequence (denote again by $\{u^m\}$) with weak limit u . In particular, for every $j \in \mathbf{Z}^d$, $u_j^m \rightarrow u_j$.

Denote $g^m = cu^m + \varphi'(u^m)$. By construction $\|g^m\|_{l^2} \rightarrow 0$. Apply P_l to it and solve for $P_l(u^m)$. We get (after some constant renaming)

$$P_l u^m = C P_l g^m + C_1 P_l(\varphi'(u^m)). \quad (9)$$

We first need an estimate for $\|P_l(\varphi'(u^m))\|_{l^2}$. Denote

$z(t) = (|e^{it\Delta_d}u^m|^2 e^{it\Delta_d}u^m) \in l^2(\mathbf{Z}^d)$. We get by (8) and Lemma 2

$$\begin{aligned} \|P_l(\varphi'(u^m))\|_{l^2} &\leq \sup_{0 \leq t \leq 1} \|P_l(e^{-it\Delta_d}(z(t)))\|_{l^2} \leq C_N \sup_{0 \leq t \leq 1} \left(2^{-lN} \|z(t)\|_{l^2} \right. \\ &\quad \left. + \|P_{l-2,l+2}z(t)\|_{l^2} \right). \end{aligned}$$

But

$$\|z(t)\|_{l^2} \leq C \|e^{it\Delta_d}u^m\|_{l^6}^3 \leq C \|e^{it\Delta_d}u^m\|_{l^2}^3 = C \|u^m\|_{l^2}^3,$$

while by Corollary 1

$$\|P_{l-2,l+2}z(t)\|_{l^2} \leq C \left(2^{-lN} \|u^m\|_{l^2}^3 + \|P_{l-4,l+4}u^m\|_{l^2}^3 \right).$$

Thus, taking l^2 norms in (9) yields

$$\begin{aligned}\|P_l u^m\|_{l^2} &\leq C \|P_l g^m\|_{l^2} + C_1 \|P_l(\varphi'(u^m))\|_{l^2} \\ &\leq C \|P_l g^m\|_{l^2} + C_N 2^{-lN} \|u^m\|_{l^2}^3 + C \|P_{l-4, l+4} u^m\|_{l^2}^3 \\ &\leq C \|g^m\|_{l^2} + C_N \lambda^3 2^{-lN} + C \|P_{l-4, l+4} u^m\|_{l^2}^3.\end{aligned}$$

In short, we have the estimate

$$\|P_l u^m\|_{l^2} \leq C \|g^m\|_{l^2} + C_N \lambda^3 2^{-lN} + C \|P_{l-4, l+4} u^m\|_{l^2}^3. \quad (10)$$

The right-hand side clearly has a limit as $m \rightarrow \infty$. Indeed, by construction $\|g^m\|_{l^2} \rightarrow 0$ and by the weak convergence: $u_j^m \rightarrow u_j$, whence $\|P_{l-4, l+4} u^m\|_{l^2} \rightarrow \|P_{l-4, l+4} u\|_{l^2}$ (for a fixed l !). We get

$$\limsup_m \|P_l u^m\|_{l^2} \leq C_N \lambda^3 2^{-lN} + C \|P_{l-4, l+4} u\|_{l^2}^3. \quad (11)$$

The quantity to consider by Lemma 3 is $\lim_l \limsup_m \|P_{\geq l} u^m\|_{l^2}$. So we estimate by using (11) with $N = 1$

$$\begin{aligned}\limsup_m \|P_{\geq l} u^m\|_{l^2} &\leq \sum_{j=0}^{\infty} \limsup_m \|P_{l+j} u^m\|_{l^2} \\ &\leq C \lambda^3 \sum_{j=0}^{\infty} 2^{-l-j} + C \sum_{j=0}^{\infty} \|P_{l+j-4, l+j+4} u\|_{l^2}^3 \\ &\leq C \lambda^3 2^{-l} + C_1 \left(\sum_{s \geq l-4} \|P_s u\|_{l^2}^3 \right) \\ &\leq C \lambda^3 2^{-l} + C_1 \left(\sum_{s \geq l-4} \|P_s u\|_{l^2}^2 \right)^{3/2} = C \lambda^3 2^{-l} + C_2 \|P_{\geq l-4} u\|_{l^2}^3.\end{aligned}$$

Take $\limsup_l$ on both sides of the resulting inequality and taking into account that for a fixed $u \in l^2$, the tail $P_{\geq l} u \rightarrow 0$ in l^2 sense (i.e. $\|P_{\geq l} u\|_{l^2} \rightarrow 0$)

$$\lim_l \limsup_m \|P_{\geq l} u^m\|_{l^2} \leq C_2 \lim_l \|P_{\geq l-4} u\|_{l^2}^3 = 0.$$

By Lemma 3, we conclude that $\{u^m\}$ is a precompact sequence and the existence part of Theorem 1 is established.

3.2. The Solution u is Spatially Well-localized and Orbitally Stable

To show that $|u_m| \leq C_N(1 + |m|)^{-N}$, it suffices to show that $\|P_l u\|_{l^2} \leq C_N 2^{-lN}$ for every integer N and for some constant C_N (which may also depend on λ).

This is done by taking the limit in (11) as $m \rightarrow \infty$. We have (even without the l^2 convergence $u^{m_k} \rightarrow u$ established in the previous section)

$$\|P_l u\|_{l^2} \leq C_N \lambda^3 2^{-lN} + C \|P_{l-4, l+4} u\|_{l^2}^3.$$

Denote $a_l := \|P_l u\|_{l^2}$. It follows that for every $\delta > 0$, there exists a bounded set $A_\delta \subset \mathbb{N}$, so that whenever $l \in \mathbb{N} \setminus A_\delta$, $\|P_{l-4, l+4} u\|_{l^2} \leq \delta$. Thus, it follows that for every $l \in \mathbb{N} \setminus A_\delta$

$$a_l \leq C_N \lambda^3 2^{-lN} + C \delta^2 (a_{l-4} + \dots + a_{l+4}).$$

Take the constant $B_N = 2(C_N + \lambda^{-2} \max_{l \in A_\delta} 2^{lN})$. Since $\max_{l \in A_\delta} a_l \leq \lambda \leq B_N \lambda^3 2^{-lN}/2$, we actually have that

$$a_l \leq B_N \lambda^3 2^{-lN} + C \delta^2 (a_{l-4} + \dots + a_{l+4})$$

is satisfied for all $l \geq 0$. At this point we need the following lemma

Lemma 5. *Let $N > 0$. Then there exists a constant $\kappa_0 = \kappa_0(N)$, so that whenever $0 < \kappa < \kappa_0$, the sequence $a_l \in l^2$ and $0 < a_l \leq C 2^{-lN} + \kappa(a_{l-4} + \dots + a_{l+4})$, for all positive integers l , one has*

$$a_k \leq C_{N, \kappa_0} (1 + \|\{a_l\}\|_{l^2}) 2^{-kN},$$

for all $k > 0$.

This is Lemma 9 in [19]. From the lemma, we have for all $k > 0$

$$a_k \leq C_N (1 + \|\{a_l\}\|_{l^2}) 2^{-kN} = C_N (1 + \lambda) 2^{-kN}.$$

It follows that if one chooses an appropriate $\delta = \delta(N)$, we have the bound

$$a_k \leq D_{N, \lambda} 2^{-kN}$$

and we have established the spatial localization property.

The orbital stability of a constrained minimizer of the Hamiltonian is classical and easy to show property. The solution is orbitally stable in the l^2 -sense and in the class of ground states. That is if S_λ is the set of ground states whose l^2 -norm is λ , then $v_g(z) = e^{i\omega z} v_*$, where v_* is the minimizer is an orbitally stable solution of the averaged equation. In particular, if $v_k(0)$ is a sequence of initial data for the averaged equation with

$$\inf_{S_\lambda} \|v_k(0) - v_g(0)\|_{l^2} \leq \varepsilon,$$

then

$$\inf_{S_\lambda} \|v_k(z) - v_g(z)\|_{l^2} \leq \varepsilon$$

for all $z \geq 0$. See, for example [16] for the proof in the case of diffraction managed NLS.

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