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Asymptotic smoothing and attractors for the generalized Benjamin–Bona–Mahony equation on $\mathbf{R}^3$

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Abstract

We study the asymptotic behavior of the solutions of the Benjamin–Bona–Mahony equation defined on $\mathbf{R}^3$. We first provide a sufficient condition to verify the asymptotic compactness of an evolution equation defined in an unbounded domain, which involves the Littlewood–Paley projection operators. We then prove the existence of an attractor for the Benjamin–Bona–Mahony equation in the phase space $H^1(\mathbf{R}^3)$ by showing the solutions are point dissipative and asymptotic compact. Finally, we establish the regularity of the attractor and show that the attractor is bounded in $H^2(\mathbf{R}^3)$.

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1. Introduction and main results

In this paper, we are interested in the asymptotic behavior of the solutions of the following Benjamin–Bona–Mahony (BBM) equation defined on $\mathbf{R}^3$:

$$u_t - \Delta u_t - v\Delta u + \operatorname{div}(f(u)) = g, \quad (1)$$

where v is a positive constant, $g \in L^2(\mathbf{R}^3)$ and $f(u) = u + \frac{1}{2}u^2$.

The BBM equation was proposed in [12] as a model for propagation of long waves which incorporates nonlinear dispersive and dissipative effects. The existence and uniqueness of solutions for this equation were studied by many authors, see, for example, [7,8,12,14,15,17,22,35,36,43]. The decay rates of solutions were investigated in [3–5,13,34,50] and the references therein. When the equation is defined in a bounded domain, the existence and finite dimensionality of the global attractor were proved in [6,16,46,49]. The regularity of the global attractor was established in [47] when the forcing term $g \in H^k$ with $k \geq 0$, and the Gevrey regularity was proved in [18] when g is in a Gevrey class. The authors of [18] also proved the existence of two determining nodes for the one-dimensional equation with periodic boundary conditions.

The first goal of this paper is to establish the existence of attractors for the BBM equation on $\mathbf{R}^3$. In other words, we will prove the following result.

Theorem 1. *Given a ball $B = \{u \in H^1(\mathbf{R}^3) : \|u\|_{H^1} \leq R\}$. If g is small enough, then the BBM equation has an attractor $\mathcal{A}$ in $H^1(\mathbf{R}^3)$ which is a compact invariant set and attracts every bounded subset of B with respect to the norm topology of $H^1(\mathbf{R}^3)$.*

Remark. As stated in Theorem 1, the existence of the attractor for the BBM equation requires that the forcing term g satisfy a smallness condition, which depends on the size of the ball B . More precisely, for every $R > 0$, there exists $\varepsilon(R) > 0$, such that g satisfies the condition:

$$\int g \, dx = 0 \quad \text{and} \quad \|g\|_{L^2} + \int |x| |g(x)| \, dx \leq \varepsilon(R),$$

where $\varepsilon(R) \rightarrow 0$ as $R \rightarrow \infty$. For more details about the smallness condition on g , we refer the reader to Theorem 3.

Note that the domain of Eq. (1) is unbounded, which causes additional difficulty when we prove the existence of attractors because, in this case, the Sobolev embeddings are not compact. There are several methods which can be used to show the existence of attractors in standard Sobolev spaces when the equations are defined in unbounded domains. One method is to show that the weak asymptotic compactness is equivalent to the strong asymptotic compactness by an energy equation technique [28,31–33,38,42]. A second method is to decompose the solution operator into a compact part and an asymptotically small part [19–21]. A third method is to prove that the solutions are

uniformly small for large space and time variables by a cut-off function [41,48] or by a weight function [1].

Generally speaking, the energy equation method proposed by Ball (see, e.g., [10,11]) depends on the weak continuity of relevant energy functionals. For the BBM equation defined in bounded domains, such weak continuity can be obtained from the compactness of Sobolev embeddings (see, e.g., [46,49]). However, for the BBM equation defined in unbounded domains, it seems that the energy equation method is not easy to use. Neither is the decomposition method. So, in this paper, we will follow the idea to obtain the existence of attractors in unbounded domains by showing the solutions are uniformly small for large space and time variables. As we will see later, for the BBM equation, it is very difficult to establish the uniform estimates in time on $\mathbf{R}^3$. That is why we will have to borrow some results from harmonic analysis. Even in this case, the method presented in this paper only works for the three-dimensional BBM equation with small forcing term g . In other words, the following question is still open: if the BBM equation on $\mathbf{R}^n$ has an attractor when $n \neq 3$ or when g does not satisfy the smallness condition. For example in the case $n = 1$ for the equation $u_t - u_{xxt} - u_{xx} + u^2 u_x = g$ it is shown in [2] that the corresponding semigroup does not have an attractor even when $g(x) = 0$.

The second goal of this paper is to prove the regularity of the attractors. For the strongly dissipative equations like the reaction–diffusion equation, the regularity of attractors follows immediately from the regularity of the solutions. But for the weakly dissipative equations like the BBM equation, the regularity of attractors is not obvious since the weakly dissipative equations have no smoothing effect on the solutions. The regularity of attractors for weakly dissipative equations have been studied in [18,25,26,23,27,37,39,47] for the equations defined in bounded domains, and in [24,28] for the equations defined in unbounded domains. In this paper, we establish the regularity of attractors for the BBM equation on $\mathbf{R}^3$, by splitting the solution into its low-frequency part and high-frequency part which is similar to the decomposition used in [24] and [28] for the weakly damped Schrödinger equation on $\mathbf{R}^2$ and the KdV equation on $\mathbf{R}$, respectively. In this respect, we have the following result.

Theorem 2. *Let $B = \{u \in H^1(\mathbf{R}^3) : \|u\|_{H^1} \leq R\}$. Then the attractor $\mathcal{A} \subseteq B$ obtained in Theorem 1 is actually bounded in $H^2(\mathbf{R}^3)$, i.e., there exists a constant C depending only on v and g such that, $\|u\|_{H^2} \leq C$ for all $u \in \mathcal{A}$.*

This paper is organized as follows. In the next section, we recall some basic results from harmonic analysis and present a sufficient condition for precompactness of a sequence in $L^p(\mathbf{R}^n)$, which is an extension of Riesz–Rellich theorem. In Section 3, we discuss the well-posedness problem and define a continuous dynamical system in $H^1(\mathbf{R}^3)$ for the BBM equation. Section 4 is devoted to the uniform estimates of the solutions in time. Under the smallness condition on the forcing term g , we will show that the solutions are uniformly bounded in the phase space $H^1(\mathbf{R}^3)$. In Section 5, we prove the existence of attractors. We first show that the solutions are uniformly small for large time and space variables, and then establish the asymptotic compactness by verifying the compact criteria given in Section 2. In the last section, we split a solution

into its low-frequency part and high-frequency part, and show that the attractors obtained in $H^1(\mathbf{R}^3)$ are actually bounded in $H^2(\mathbf{R}^3)$.

2. Preliminaries

In this section, we discuss the precompactness of a sequence in $L^p(\mathbf{R}^n)$. We first recall some basic results from harmonic analysis, and then generalize Riesz–Rellich criteria for compactness by using the Littlewood–Paley projection operators.

2.1. Fourier transform, Littlewood–Paley projections and commutators

Denote by $\mathcal{S}$ the Schwartz class. Then given $f \in \mathcal{S}$, the Fourier transform of f is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Fix an *even* function φ in $C_0^\infty(\mathbf{R}^n)$, so that the support of φ is in the annulus $\frac{1}{2} \leq |\xi| \leq 2$ and

$$\sum_{k=-\infty}^{\infty} \varphi(2^{-k}x) = 1 \quad \text{for all } x \neq 0.$$

Define the operator P_δ via $\widehat{P_\delta f}(\xi) = \varphi(\delta^{-1}\xi) \hat{f}(\xi)$. Observe that P_δ essentially restricts the Fourier support of the function f to the annulus $\delta/2 \leq |\xi| \leq 2\delta$ and $\sum_{\delta \text{ dyadic}} P_\delta = 1$. Sometimes we will denote $P_\delta u$ simply by u_δ and call this the *Littlewood–Paley projection operator* at frequency δ . We will also make use of the operators $P_{<\delta} := \sum_{\delta' < \delta} P_{\delta'}$, etc. Note that the kernel form of such operators is given by

$$P_\delta f = \delta^n \widehat{\varphi}(\delta \cdot) * f$$

and thus, since $\|\delta^n \widehat{\varphi}(\delta \cdot)\|_{L^1} \lesssim 1$, we have that $P_\delta : L^p \rightarrow L^p$ for all $1 \leq p \leq \infty$. Here and after, by $f \lesssim g$, we mean that there is an absolute constant C such that $f \leq Cg$. By $f \sim g$, we mean $f \lesssim g$ and $g \lesssim f$.

Since the function φ is even, we see the kernels of P_δ are real valued. The Bernstein inequality, then is stated as follows: for all $1 \leq p \leq q \leq \infty$:

$$\|P_\delta f\|_{L^q(\mathbf{R}^n)} \leq C_{p,q} \delta^{n(1/p-1/q)} \|f\|_{L^p(\mathbf{R}^n)},$$

where $C_{p,q}$ is a constant depending only on p and q . Throughout this paper, we denote by $\|\cdot\|_{L^p(\mathbf{R}^n)}$ the norm of $L^p(\mathbf{R}^n)$ and $\|\cdot\|$ the norm of $L^2(\mathbf{R}^n)$. Next, we concentrate on the properties of certain commutators involving P_δ and multiplication by (smooth)

functions, that is, we are interested in the L^p mapping properties of operators in the form $[P_\delta, \psi]g := P_\delta(\psi g) - \psi P_\delta g$. We shall need the following result, which is standard. We reproduce here its elegant proof, due to Tao [44].

Lemma 1. *Let m be a Schwartz function and $Q_\delta : \widehat{Q_\delta f}(\xi) = m(\xi/\delta)\hat{f}(\xi)$. Let f be a Lipschitz function. Then for every $1 \leq p \leq \infty$:*

$$\|[Q_\delta, f]g\|_{L^p} \leq \delta^{-1} \|f\|_{\text{Lip}} \left(\int |y| |\widehat{m}(y)| dy \right) \|g\|_{L^p}, \quad (2)$$

$$\|[Q_\delta, f]\nabla g\|_{L^p} \leq \|f\|_{\text{Lip}} \left(\int (1 + |y|)(|\widehat{m}(y)| + |\nabla \widehat{m}(y)|) dy \right) \|g\|_{L^p}. \quad (3)$$

In particular, for P_δ , we have the estimates

$$\|[P_\delta, f]g\|_{L^p} \lesssim \delta^{-1} \|f\|_{\text{Lip}} \|g\|_{L^p}, \quad (4)$$

$$\|[P_\delta, f]\nabla g\|_{L^p} \lesssim \|f\|_{\text{Lip}} \|g\|_{L^p}. \quad (5)$$

Proof. By the definition, we have

$$[Q_\delta, f]g(x) = \delta^n \int \check{m}(\delta y)(f(x-y) - f(x))g(x-y) dy.$$

By the Lipschitz continuity, we obtain

$$|[Q_\delta, f]g(x)| \leq \|f\|_{\text{Lip}} \delta^n \int |\check{m}(\delta y)| |y| |g(x-y)| dy.$$

Take L^p norms in x

$$\|[Q_\delta, f]g\|_{L^p} \lesssim \|f\|_{\text{Lip}} \|g\|_{L^p} \delta^n \int |\check{m}(\delta y)| |y| dy \lesssim \delta^{-1} \|f\|_{\text{Lip}} \|g\|_{L^p},$$

which is (2). For the proof of (3), one has to do an extra integration by parts and apply the same estimates. $\square$

We also have the following useful generalizations.

Corollary 1. *For the operator $\tilde{P}_\delta^{(l)} := P_\delta |\nabla|^l$ with multiplier $\varphi(\xi/\delta)|\xi|^l$, there is*

$$\|\tilde{P}_\delta^{(l)} f\|_{L^p} \lesssim \delta^l \|f\|_{L^p},$$

$$\|[\tilde{P}_\delta^{(l)}, f]g\|_{L^p} \lesssim \delta^{l-1} \|f\|_{\text{Lip}} \|g\|_{L^p},$$

$$\|[\tilde{P}_\delta^{(l)}, f]\nabla g\|_{L^p} \lesssim \delta^l \|f\|_{\text{Lip}} \|g\|_{L^p}.$$

In fact, all one needs to do in order to prove Corollary 1 is to observe that the multiplier of $\tilde{P}_\delta^l$ is in the form $\delta^l \tilde{m}(\xi/\delta)$ and apply the previous results.

The next result takes a closer look at the relations between spatial cut-off of the Littlewood–Paley projection and its gradient. Before we state our result, define the Riesz projections R_j as the operators with multipliers $-i\xi_j/|\xi|$. Observe that $|\nabla| = \sum_j R_j \partial_j = \langle R, \nabla \rangle$. With a slight abuse of notation we write $|\nabla| = R\nabla$.

Lemma 2. *Let $\delta, N > 0$ and fix $1 < p < \infty$. Let $\psi(x) = h^p(x)$, where h is a positive Lipschitz function with a Lipschitz constant $\|h\|_{\text{Lip}}$. Then there exists a constant $C = C(n, \|h\|_{\text{Lip}})$, so that*

$$\begin{aligned} & \delta \left(\int_{\mathbf{R}^n} |P_\delta u|^p \psi(x/N) dx \right)^{1/p} \\ & \leq C \left(\int_{\mathbf{R}^n} |\nabla P_\delta u|^p \psi(x/N) dx \right)^{1/p} + CN^{-1} \|P_\delta u\|_{L^p}. \end{aligned} \quad (6)$$

In particular,

$$\delta \left(\int |P_\delta u|^p \psi(x/N) dx \right)^{1/p} \leq C \left(\int |\nabla P_\delta u|^p \psi(x/N) dx \right)^{1/p} + CN^{-1} \|u\|_{L^p}. \quad (7)$$

Proof. Define first $\tilde{P}_\delta$ to be some Littlewood–Paley projection with a multiplier supported in the ball $\delta/10 \leq |\xi| \leq 10\delta$ and so that $\tilde{P}_\delta P_\delta = P_\delta$. This can be achieved by requiring that the multiplier of $\tilde{P}_\delta$ is identically one on the support of the multiplier of P_δ .

Write

$$P_\delta u = \tilde{P}_\delta P_\delta u = (R \tilde{P}_\delta |\nabla|^{-1}) [P_\delta \nabla u] = \tilde{P}_\delta^{(-1)} [P_\delta \nabla u],$$

where $\tilde{P}_\delta^{(-1)} = R \tilde{P}_\delta |\nabla|^{-1}$ which is similar to the operators defined in Corollary 1. We have

$$\begin{aligned} & \delta \left(\int |P_\delta u|^p \psi(x/N) dx \right)^{1/p} \\ & = \delta \| (P_\delta u) h(\cdot/N) \|_{L^p} = \delta \| (\tilde{P}_\delta^{(-1)} [P_\delta \nabla u]) h(\cdot/N) \|_{L^p} \\ & \leq \delta \| \tilde{P}_\delta^{(-1)} [(P_\delta \nabla u) h(\cdot/N)] \|_{L^p} + \delta \| [\tilde{P}_\delta^{(-1)}, h(\cdot/N)] \nabla P_\delta u \|_{L^p} \\ & \lesssim \| (P_\delta \nabla u) h(\cdot/N) \|_{L^p} + \| P_\delta u \|_{L^p} / N, \end{aligned}$$

where in the last line we have used Corollary 1 with $l = -1$ and $\|h(\cdot/N)\|_{\text{Lip}} \sim N^{-1}$. $\square$

By summing (6) over all dyadic $\tilde{\delta} \geq \delta$ and taking into account $\|P_{\tilde{\delta}} \nabla u\|_{L^p} \sim \tilde{\delta} \|P_{\tilde{\delta}} u\|_{L^p}$, we obtain

Corollary 2. For all $1 < p < \infty$ and all $\delta > 0$,

$$\begin{aligned} & \delta \left(\int |P_{\geq \delta} u|^p \psi(x/N) dx \right)^{1/p} \\ & \leq C \left(\int |\nabla P_{\geq \delta} u|^p \psi(x/N) dx \right)^{1/p} + \frac{C}{N\delta} \|P_{\geq \delta} \nabla u\|_{L^p}. \end{aligned} \quad (8)$$

2.2. Riesz–Relich criteria for compactness in L^p

We now discuss the compactness of a set in $L^p(\mathbf{R}^n)$. Recall the following theorem due to Riesz (see [40, Theorem XIII.66, p. 248]).

Proposition 1. Let $S \subseteq L^p(\mathbf{R}^n)$ with $1 \leq p < \infty$. Then S is precompact in $L^p(\mathbf{R}^n)$ if and only if the following conditions are satisfied:

- (1) S is bounded in $L^p(\mathbf{R}^n)$;
- (2) $f \rightarrow 0$ in L^p sense at infinity uniformly in S , i.e., for any ε , there is a bounded set $K \subset \mathbf{R}^n$ so that for all $f \in S$:

$$\int_{\mathbf{R}^n \setminus K} |f(x)|^p dx \leq \varepsilon^p;$$

- (3) $f(\cdot - y) \rightarrow f$ uniformly in S as $y \rightarrow 0$, i.e., for any ε , there is δ so that $f \in S$ and $|y| < \delta$ imply that

$$\int_{\mathbf{R}^n} |f(x - y) - f(x)|^p dx \leq \varepsilon^p.$$

We note that in a bounded domain D , the condition for compactness of a sequence in $L^p(D)$ reduces to (1) and (3), since (2) is obviously satisfied.

In terms of the Littlewood–Paley projections, the following compactness result is not unexpected.

Proposition 2. Assume $\{f_n\}_{n=1}^\infty$ is bounded in $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$. Then $\{f_n\}_{n=1}^\infty$ is precompact in $L^p(\mathbf{R}^n)$ if and only if for every $\delta_0 > 0$ the sequence $\{P_{\geq \delta_0} f_n\}_{n=1}^\infty$ is precompact.

Proof. The necessity part is clear. For the sufficiency, construct a convergent subsequence of $\{P_{\geq 1} f_n\}$, say $\{f_{n_k^1}\}$. From $\{f_{n_k^1}\}$, extract a further subsequence $\{f_{n_k^2}\}$, so that $\{P_{\geq 1/2} f_{n_k^2}\}$ is convergent and continue in that fashion. By the boundedness of $P_{\geq \delta}$ on

L^p and the construction of the sequences, the limits $h_s := \lim_k P_{\geq 2^{-s}} f_{n_k^s}$ will satisfy the consistency condition $P_{\geq 2^{-s}} h_{s_1} = P_{\geq 2^{-s}} h_{s_2}$, whenever $s_1, s_2 > s + 2$. By the uniform boundedness, we also have $\sup_s \|h_s\|_{L^p} < \infty$. This allows us to define a function $h \in L^p(\mathbf{R}^n)$ with $P_{\geq 2^{-s}} h = P_{\geq 2^{-s}} h_{s_1}$ for every $s_1 > s + 2$. It is clear now that the diagonal sequence $\{f_{n_s^s}\}$ will be such that for each $\delta > 0$, $\lim_s P_{\geq \delta} f_{n_s^s} = P_{\geq \delta} h$. It follows that $\lim_s f_{n_s^s} = h$ and thus the precompactness of $\{f_n\}_{n=1}^\infty$. $\square$

In the special and important case $p = 2$, Relich [40, Theorem XIII.65] has shown that instead of checking (the hard to verify) condition (3) in Proposition 1, one may equivalently check that the L^2 mass of the Fourier transform on the complements of large balls tends to zero. We have a variant, which suits our purposes better, but the next proposition is essentially equivalent to Theorem XIII.65 in [40].

Proposition 3. Assume $\{f_n\}_{n=1}^\infty$ is bounded in $L^2(\mathbf{R}^n)$. Then $\{f_n\}_{n=1}^\infty$ is precompact in $L^2(\mathbf{R}^n)$ if and only if

- $\lim_{N \rightarrow \infty} \sup_n \|f_n\|_{L^2(|x| \geq N)} = 0$
- $\lim_{N \rightarrow \infty} \sup_n \|P_{\geq N} f_n\|_{L^2} = 0$

Proof. The necessity part is trivial. We check the sufficiency part. By Proposition 1, we need only check that for uniform bounded sequence $\{f_n\}$, $\limsup_{y \rightarrow 0} \sup_n \|f_n(\cdot - y) - f_n(\cdot)\|_{L^2} = 0$, provided $\lim_{N \rightarrow \infty} \sup_n \|P_{\geq N} f_n\|_{L^2} = 0$. To that end, take $\varepsilon > 0$ and select $M = M(\varepsilon)$ so that

$$\sup_n \int_{|\xi| > M} |\widehat{f_n}(\xi)|^2 d\xi \leq \varepsilon^2.$$

Choose $\delta = \varepsilon / (M^2 \sup_n \|f_n\|_{L^2})$. For every $|h| \leq \delta$, we have by the Plancherel's theorem

$$\begin{aligned} \sup_n \|f_n(\cdot - h) - f_n(\cdot)\|_{L^2}^2 &\leq \sup_n \int |e^{2\pi i h \xi} - 1|^2 |\widehat{f_n}(\xi)|^2 d\xi \\ &\lesssim \sup_n \int_{|\xi| > M} |\widehat{f_n}(\xi)|^2 d\xi \\ &\quad + h^2 \sup_n \int_{|\xi| < M} |\xi|^4 |\widehat{f_n}(\xi)|^2 d\xi \\ &\lesssim \varepsilon^2 + h^2 M^4 \sup_n \|f_n\|_{L^2}^2 \leq 2\varepsilon^2. \end{aligned}$$

The proof is complete. $\square$

The next lemma, which somewhat improves the standard Riesz–Relich criteria is crucial in our arguments later on.

Lemma 3. Let $\{f_n\} \subset L^2(\mathbf{R}^n)$ be uniformly bounded and

$$\limsup_{N \rightarrow \infty} \sup_n \|P_{\geq N} f_n\|_{L^2(|x| \geq N)} = h(\delta), \quad \lim_{\delta \rightarrow 0} h(\delta) = 0, \quad (9)$$

$$\lim_{N \rightarrow \infty} \sup_n \|P_{\geq N} f_n\|_{L^2} = 0. \quad (10)$$

Then the sequence $\{f_n\}_{n=1}^\infty$ is precompact in $L^2(\mathbf{R}^n)$.

Proof. By the uniform boundedness of $\{f_n\}$, there is $f \in L^2(\mathbf{R}^n)$ such that, up to a subsequence (which we call again f_n), f_n is weakly convergent to f . We need to show that there exists a subsequence of $\{f_n\}$ which is strongly convergent to f in $L^2(\mathbf{R}^n)$.

To that end, we first prove by induction that there exist subsequences $\{n_k^s\}$ and functions $g_k \in L^2(\mathbf{R}^n)$ so that $\{n_{k+1}^s\}_s \subset \{n_k^s\}_s$ and $\limsup_{s \rightarrow \infty} \|P_{\geq 2^{-k}}(f_{n_k^s}) - g_k\|_{L^2} \leq 2h(2^{-k})$.

The subsequences can be chosen as follows. By (9), choose N_k , so that

$$\sup_n \|P_{\geq 2^{-k}} f_n\|_{L^2(|x| > N_k)} \leq 2h(2^{-k}). \quad (11)$$

We now prove that when restricted in the bounded set $\{x \in \mathbf{R}^n : |x| \leq N_k\}$, the sequence $\{P_{\geq 2^{-k}} f_{n_{k-1}^s}\}$ is precompact, which follows from Riesz–Rellich Theorem in bounded domains provided condition (3) in Proposition 1 is verified. Note that condition (3) in Proposition 1 is implied by (10). Indeed we have

$$\begin{aligned} & \sup_n \|P_{\geq 2^{-k}} f_n(\cdot - y) - P_{\geq 2^{-k}} f_n(\cdot)\|_{L^2(|x| \leq N_k)} \\ & \leq \sup_n \|P_{\geq 2^{-k}} f_n(\cdot - y) - P_{\geq 2^{-k}} f_n(\cdot)\|_{L^2} \\ & \leq \sup_n \|f_n(\cdot - y) - f_n(\cdot)\|_{L^2}. \end{aligned}$$

By (10) and the uniform boundedness of f_n , the last expression converges to zero as $y \rightarrow 0$, see the proof of Proposition 3. Therefore we find

$$\lim_{y \rightarrow 0} \sup_n \|P_{\geq 2^{-k}} f_n(\cdot - y) - P_{\geq 2^{-k}} f_n(\cdot)\|_{L^2(|x| \leq N_k)} = 0,$$

which along with the boundedness of $\{f_n\}$ show that the sequence $\{P_{\geq 2^{-k}} f_{n_{k-1}^s}\}$ satisfies all conditions in Proposition 1 when restricted in the bounded set $\{x \in \mathbf{R}^n : |x| \leq N_k\}$. Then by Riesz–Rellich Theorem, there is $g_k \in L^2(\mathbf{R}^n)$ with $\text{supp } g_k \subseteq \{x \in \mathbf{R}^n : |x| \leq N_k\}$, and a subsequence $\{P_{\geq 2^{-k}} f_{n_k^s}\}$ of $\{P_{\geq 2^{-k}} f_{n_{k-1}^s}\}$ such that $\{P_{\geq 2^{-k}} f_{n_k^s}\} \rightarrow g_k$ in $L^2(|x| \leq N_k)$.

From (11)

$$\begin{aligned} \limsup_s \left\| P_{\geq 2^{-k}} f_{n_k^s} - g_k \right\|_{L^2} &\leq \lim_s \left\| P_{\geq 2^{-k}} f_{n_k^s} - g_k \right\|_{L^2(|x| \leq N_k)} \\ &\quad + \limsup_s \left\| P_{\geq 2^{-k}} f_{n_k^s} \right\|_{L^2(|x| \geq N_k)} \leq 2h(2^{-k}). \end{aligned} \quad (12)$$

Next, we show that $g_k \rightarrow f$ in $L^2(\mathbf{R}^n)$. Note that for each k , there is a function $\psi_k \in L^2(\mathbf{R}^n)$ with $\|\psi_k\|_{L^2} = 1$ so that

$$\left\| P_{\geq 2^{-k}} f - g_k \right\|_{L^2} = \langle P_{\geq 2^{-k}} f - g_k, \psi_k \rangle.$$

Then by the weak convergence of f_n to f , we get

$$\begin{aligned} &\left\| P_{\geq 2^{-k}} f - g_k \right\|_{L^2} \\ &= \langle P_{\geq 2^{-k}} f - g_k, \psi_k \rangle = \langle P_{\geq 2^{-k}} f - P_{\geq 2^{-k}} f_{n_k^s}, \psi_k \rangle + \langle P_{\geq 2^{-k}} f_{n_k^s} - g_k, \psi_k \rangle \\ &\leq \limsup_s \langle P_{\geq 2^{-k}} f - P_{\geq 2^{-k}} f_{n_k^s}, \psi_k \rangle + 2h(2^{-k}) = 2h(2^{-k}), \end{aligned}$$

which implies $g_k \rightarrow f$ in $L^2(\mathbf{R}^n)$. We now prove that for each $\delta_0 > 0$,

$$P_{\geq \delta_0} f_{n_k^{s_k}} \rightarrow P_{\geq \delta_0} f \quad \text{in } L^2(\mathbf{R}^n), \quad k \rightarrow \infty. \quad (13)$$

To this end, we choose s_k , so that $\left\| P_{\geq 2^{-k}} f_{n_k^{s_k}} - g_k \right\|_{L^2} \leq 4h(2^{-k})$. Then for large k with $2^{-k} \ll \delta_0$ we have $P_{\geq \delta_0} P_{\geq 2^{-k}} = P_{\geq \delta_0}$, whence

$$\begin{aligned} \left\| P_{\geq \delta_0} f_{n_k^{s_k}} - P_{\geq \delta_0} f \right\|_{L^2} &= \left\| P_{\geq \delta_0} P_{\geq 2^{-k}} f_{n_k^{s_k}} - P_{\geq \delta_0} f \right\|_{L^2} \leq \left\| P_{\geq 2^{-k}} f_{n_k^{s_k}} - f \right\|_{L^2} \\ &\leq \left\| P_{\geq 2^{-k}} f_{n_k^{s_k}} - g_k \right\|_{L^2} + \|g_k - f\|_{L^2} \\ &\leq 4h(2^{-k}) + \|g_k - f\|_{L^2} \rightarrow 0, \quad \text{as } k \rightarrow \infty, \end{aligned}$$

which implies (13). By (13) and Proposition 2, we know the sequence $\{f_n\}$ is precompact in $L^2(\mathbf{R}^n)$. The proof is complete. $\square$

As an immediate consequence of Lemma 3, we have the following results, whose proof is omitted.

Lemma 4. *Let $\{f_n\} \subset L^2(\mathbf{R}^n)$ be uniformly bounded and*

$$\limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| P_{\geq \delta} f_n \right\|_{L^2(|x| \geq N)} = h(\delta), \quad \lim_{\delta \rightarrow 0} h(\delta) = 0, \quad (14)$$

$$\limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \|P_{\geq N} f_n\|_{L^2} = 0. \quad (15)$$

Then the sequence $\{f_n\}_{n=1}^{\infty}$ is precompact in $L^2(\mathbf{R}^n)$.

3. Well-posedness

In this section, we first discuss the well-posedness problem of the BBM equation on $\mathbf{R}^n$, and then define a continuous dynamical system $S(t)_{t \geq 0}$ in the phase space $H^1(\mathbf{R}^n)$.

Since $f(u) = u + \frac{1}{2}u^2$, the BBM equation can be rewritten as follows:

$$\begin{cases} u_t - v\Delta(1 - \Delta)^{-1}u + \operatorname{div}(1 - \Delta)^{-1}\left(u + \frac{1}{2}u^2\right) = (1 - \Delta)^{-1}g, \\ u(0) = u_0. \end{cases} \quad (16)$$

For the local well-posedness, we further rewrite (16) as an equivalent integral equation:

$$u(t) = u_0 + \int_0^t \left(v\Delta(1 - \Delta)^{-1}u(s) - \operatorname{div}(1 - \Delta)^{-1}\left(u + \frac{1}{2}u^2\right)(s) + (1 - \Delta)^{-1}g \right) ds.$$

Denote by

$$F(u) = v\Delta(1 - \Delta)^{-1}u - \operatorname{div}(1 - \Delta)^{-1}\left(u + \frac{1}{2}u^2\right) + (1 - \Delta)^{-1}g.$$

It is easy to show that, if $n \leq 4$, then F maps $H^1(\mathbf{R}^n)$ into itself and is locally Lipschitz continuous. More precisely, there exists a constant C such that

$$\|F(u)\|_{H^1} \leq C\|u\|_{H^1}^2 + v\|u\|_{H^1} + C\|g\|_{L^2}, \quad (17)$$

$$\|F(u) - F(v)\|_{H^1} \leq C\|u - v\|_{H^1}(1 + \max(\|u\|_{H^1}, \|v\|_{H^1})). \quad (18)$$

Then by the fixed point argument, we find that the BBM equation is locally well-posed in $H^1(\mathbf{R}^n)$ with $n \leq 4$. The next estimate shows that the local solution is actually defined for all $t \geq 0$.

Lemma 5. *Let $u_0 \in H^1(\mathbf{R}^n)$. Then for each $T > 0$, there exists a constant C depending on u_0 , T and g such that for all $t \in [0, T)$:*

$$\|u(t)\|_{H^1} \leq C.$$

Proof. Multiplying Eq. (1) by u and then integrating, we get

$$\frac{1}{2} \frac{d}{dt} \left(\|u(t)\|^2 + \|\nabla u(t)\|^2 \right) + v \|\nabla u(t)\|^2 = \int_{\mathbf{R}^n} g(x) u(x) dx,$$

which implies that

$$\frac{d}{dt} \left(\|u(t)\|^2 + \|\nabla u(t)\|^2 \right) \leq \left(\|u(t)\|^2 + \|\nabla u(t)\|^2 \right) + \|g\|^2.$$

Then Lemma 5 follows from the above and Gronwall lemma. $\square$

As a consequence of Lemma 5, we obtain the following global well-posedness result for the BBM equation (see also [7]).

Lemma 6. *Let $g \in L^2(\mathbf{R}^n)$ and $n \leq 4$. If $u_0 \in H^1(\mathbf{R}^n)$, then the BBM equation (1) has a unique global solution $u \in C([0, \infty), H^1(\mathbf{R}^n))$. Furthermore, the solutions are continuous with respect to the initial data in $H^1(\mathbf{R}^n)$.*

By Lemma 6, we can define a continuous dynamical system $S(t)_{t \geq 0}$ such that for each $t \geq 0$, $S(t)u_0 = u(t)$, where $u(t)$ is the solution of the BBM equation. In the remaining sections of the paper, we will discuss the asymptotic behavior of $S(t)_{t \geq 0}$ when $t \rightarrow \infty$.

4. Uniform estimates in time

In this section, we derive the uniform estimates of the solutions when $t \rightarrow \infty$. As we will see later, the biggest obstacle in proving the uniform boundedness of the orbits comes from the possible concentration of L^2 mass around low frequencies. In order to prevent the buildup of low frequency mass, we impose the following technical condition on g : there exists $\mu > 0$ and a constant G , so that for every $\delta < 1$:

$$\|P_\delta g\|_{L^2} \leq G\delta^{2+\mu}. \quad (19)$$

Remark. The following argument shows that (19) is satisfied for sufficiently decaying g , with $\int g = 0$ (i.e. $\hat{g}(0) = 0$). Indeed, we have

$$\begin{aligned} \|P_\delta g\|_{L^2}^2 &= \int_{|\xi| \sim \delta} |\hat{g}(\xi)|^2 d\xi = \int_{|\xi| \sim \delta} |\hat{g}(\xi) - \hat{g}(0)|^2 d\xi \leq \int_{|\xi| \sim \delta} \left| \int_0^1 |\nabla_\xi \hat{g}(z\xi)| dz \right|^2 |\xi|^2 d\xi \\ &\lesssim \delta^{n+2} \sup_{|\xi| \leq 1} |\nabla_\xi \hat{g}(\xi)|^2 \lesssim \delta^{n+2} \|\widehat{x \cdot g}\|_{L^\infty} \lesssim \delta^5 \left(\int |x| |g(x)| dx \right)^2. \end{aligned}$$

As a consequence, (19) is satisfied for all g with $\int g = 0$ and $\int |x| |g(x)| dx < \infty$.

In order to establish the uniform bounds of the solutions, we need to decompose the solution as a sum of its low frequencies and high frequencies, and estimate the low frequencies and high frequencies separately. We first estimate the low frequencies in the next section.

4.1. Estimates on low frequencies of solutions

In the sequel, we derive the uniform estimates on the low frequencies of the solutions. For convenience, we introduce the following functionals: for $u \in H^1(\mathbf{R}^n)$,

$$I(u) = \int (u^2 + |\nabla u|^2) dx,$$

and for each $\delta \leq 1$:

$$I_\delta(u) = \int (|u_\delta|^2 + |\nabla u_\delta|^2) dx.$$

If u is the solution of the BBM equation, then we define the maximal energy functionals

$$J(T) = \sup_{0 \leq t \leq T} I(u(t)), \quad (20)$$

and

$$J_\delta(T) = \sup_{0 \leq t \leq T} I_\delta(u(t)). \quad (21)$$

For the low frequencies of the solutions, we have the following estimates.

Lemma 7. Assume that (19) holds and $g \in L^2(\mathbf{R}^n)$, $n = 3$ or 4 . Then for each $\delta_0 \leq 1$, the following inequality holds:

$$J_{\leq \delta_0}(T) \leq I_{\leq \delta_0}(0) + C_v \delta_0^{n/2-1} J^{3/2}(T) + C_{v,\mu} G^2, \quad (22)$$

where G is the constant in (19), C_v depends on v , $C_{v,\mu}$ depends on v and μ .

Proof. Fix $\delta \leq \delta_0$. Then applying the Littlewood–Paley projection P_δ to (1), we find

$$\frac{1}{2} \frac{d}{dt} I_\delta + v \int |\nabla u_\delta|^2 dx + \int \left(\operatorname{div} \left(u + \frac{1}{2} u^2 \right)_\delta \right) u_\delta dx = \int g_\delta u_\delta dx.$$

We estimate the various terms arising in the above energy equation. We first have

$$v \int |\nabla u_\delta|^2 dx \geq v\delta^2/8 \int |u_\delta|^2 dx + v/2 \int |\nabla u_\delta|^2 dx \geq \frac{1}{8} v\delta^2 I_\delta(t).$$

For the next term in the energy equation, we obtain

$$\begin{aligned} \left| \int (\operatorname{div}(\frac{1}{2}u^2)_\delta) u_\delta dx \right| &\leq \frac{1}{2} \left\| \operatorname{div}(u^2)_\delta \right\|_{L^2} \|u_\delta\|_{L^2} \lesssim \delta \left\| (u^2)_\delta \right\|_{L^2} \|u_\delta\|_{L^2} \\ &\lesssim \delta^{n/2+1} \left\| (u^2)_\delta \right\|_{L^1} \|u_\delta\|_{L^2} \lesssim \delta^{n/2+1} \|u\|_{L^2}^2 \|u_\delta\|_{L^2} \lesssim \delta^{n/2+1} I^{3/2}(t). \end{aligned}$$

In the string of inequalities above, we have used the Cauchy–Schwartz inequality, the fact that $\left\| \operatorname{div} \chi_{\{\delta/2 \leq |\xi| \leq 2\delta\}} \right\|_{L^2 \rightarrow L^2} \lesssim \delta$ and the Bernstein inequality. For the term arising from the forcing term, by (19) we get

$$\left| \int g_\delta u_\delta dx \right| \leq \|g_\delta\|_{L^2} \|u_\delta\|_{L^2} \leq G\delta^{2+\mu} \sqrt{I_\delta(t)} \leq 4G^2 v^{-1} \delta^{2+2\mu} + \frac{v\delta^2}{16} I_\delta(t).$$

Putting everything together yields the differential inequality:

$$\frac{1}{2} \frac{d}{dt} I_\delta + \frac{1}{16} v\delta^2 I_\delta \leq C\delta^{n/2+1} I^{3/2}(t) + 4G^2 v^{-1} \delta^{2+2\mu}.$$

Multiplying both sides by $e^{v\delta^2 t/8}$ and integrating in time yields

$$I_\delta(t) \leq e^{-v\delta^2 t/8} I_\delta(0) + C v^{-1} \delta^{n/2-1} \sup_{0 \leq t' \leq t} I^{3/2}(t') + 64G^2 v^{-2} \delta^{2\mu}.$$

By (20) and (21), it follows from the above that

$$J_\delta(T) \leq I_\delta(0) + C v^{-1} \delta^{n/2-1} J^{3/2}(T) + 64\delta^{2\mu} v^{-2} G^2.$$

Taking a dyadic sum on both sides of the inequality in $\delta \leq \delta_0$, then Lemma 7 follows. $\square$

4.2. Estimates on high frequencies of solutions

In the sequel, we derive the uniform estimates on the high frequencies of the solutions. More precisely, we prove the following inequality.

Lemma 8. Let $g \in L^2(\mathbf{R}^n)$, $n = 3$ or 4 . Then for each $\delta_0 \geq 1$, the following inequality holds:

$$J_{\geq \delta_0}(T) \leq I_{\geq \delta_0}(0) + C_v \delta_0^{n/2-1} J^{3/2}(T) + C_v \|g\|^2 \delta_0^{-4}, \quad (23)$$

where C_v is a constant depending on v .

Proof. Take the Littlewood–Paley projection $P_{\geq \delta_0}$ on both sides of (1). We find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} I_{\geq \delta_0}(t) + v \int |\nabla u_{\geq \delta_0}|^2 dx + \int \operatorname{div} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta_0} u_{\geq \delta_0} dx \\ = \int g_{\geq \delta_0} u_{\geq \delta_0} dx. \end{aligned} \quad (24)$$

By Fourier support considerations, one has

$$v \int |\nabla u_{\geq \delta_0}|^2 dx \geq \frac{v \delta_0^2}{8} I_{\geq \delta_0}.$$

Since $\int \operatorname{div}(u^2) u dx = 0$, one has by the Bernstein inequality

$$\begin{aligned} \left| \int \operatorname{div} \left(\frac{1}{2} u^2 \right)_{\geq \delta_0} u_{\geq \delta_0} dx \right| &= \frac{1}{2} \left| \int \operatorname{div}(u^2)_{\leq \delta_0} u_{\leq \delta_0} dx \right| \\ &\leq \left\| \operatorname{div}(u^2)_{\leq \delta_0} \right\|_{L^2} \|u_{\leq \delta_0}\|_{L^2} \\ &\leq C \delta_0^{n/2+1} \left\| (u^2)_{\leq \delta_0} \right\|_{L^1} \|u_{\leq \delta_0}\|_{L^2} \leq C \delta_0^{n/2+1} I^{3/2}(t). \end{aligned}$$

Finally,

$$\left| \int g_{\geq \delta_0} v_{\geq \delta_0} dx \right| \leq \|g\|_{L^2} \|v_{\geq \delta_0}\|_{L^2} \leq \|g\|_{L^2} \sqrt{I_{\geq \delta_0}(t)} \leq 4 \|g\|_{L^2}^2 v^{-1} \delta_0^{-2} + \frac{1}{16} v \delta_0^2 I_{\geq \delta_0}(t).$$

Substituting the above estimates into (24), we get

$$\frac{d}{dt} I_{\geq \delta_0}(t) + \frac{1}{8} v \delta_0^2 I_{\geq \delta_0} \leq C \delta_0^{n/2+1} I^{3/2}(t) + 8 v^{-1} \delta_0^{-2} \|g\|_{L^2}^2,$$

which implies Lemma 8. $\square$

4.3. Uniform estimates

We are now in a position to show that the solutions are uniformly bounded in time, that is, we have the following estimates.

Lemma 9. Assume that (19) holds and $g \in L^2(\mathbf{R}^n)$, $n = 3$ or 4 . Then the solutions of the BBM equation are uniformly bounded in time for small forcing term g . More precisely, for every constant R , there exists $\varepsilon = \varepsilon(R, \nu, \mu)$, so that if $\|u_0\|_{H^1} \leq R$ and

$$\|g\|_{L^2} + G \leq \varepsilon,$$

where G is the constant in (19), then for all $t \geq 0$:

$$\|u(t)\|_{H^1} \leq 2(1 + R^2).$$

Proof. By Lemmas 7 and 8 we obtain: for every $t \geq 0$,

$$J(t) \leq I(0) + C_\nu \delta_0^{n/2-1} J^{3/2}(t) + C_{\nu,\mu} G^2 + C_\nu \|g\|_{L^2}^2 \delta_0^{-4}. \quad (25)$$

Let

$$\delta_0 = \frac{1}{1 + 8C_\nu^2(1 + R^2)}, \quad \tilde{\delta} = \frac{1}{8C_\nu^2(1 + R^2)}.$$

Then we see $\delta_0 \leq 1$ and $\delta_0 \leq \tilde{\delta}$. Take ε small enough such that

$$\varepsilon^2 \leq \frac{1}{2(C_{\nu,\mu} + C_\nu \delta_0^{-4})}.$$

Next, we prove $J(t) \leq 2(1 + R^2)$ for all $t \geq 0$. To this end, we denote by

$$T = \sup\{t \geq 0 : J(t) \leq 2(1 + R^2)\}.$$

Clearly, $T > 0$. We now show that $T = \infty$. If $T < \infty$, then by the definition we find $J(T) = 2(1 + R^2)$. In this case, (25) implies that

$$\begin{aligned} 2 + 2R^2 &\leq I(0) + C_\nu \delta_0^{1/2} (2(1 + R^2))^{3/2} + (C_{\nu,\mu} + C_\nu \delta_0^{-4}) \varepsilon^2 \\ &\leq R^2 + C_\nu \tilde{\delta}^{1/2} (2(1 + R^2))^{3/2} + \frac{1}{2} \leq \frac{3}{2} + 2R^2. \end{aligned}$$

It then follows that $\frac{1}{2} \leq 0$, a contradiction which demonstrates $T = \infty$ and therefore $J(t) \leq 2(1 + R^2)$ for all $t \geq 0$. $\square$

5. Existence of attractors

In this section, we establish the existence of attractors for the dynamical system $S(t)_{t \geq 0}$. It is known that an attractor exists if $S(t)_{t \geq 0}$ is point dissipative and asymp-

totic compact. The point dissipativeness of $S(t)_{t \geq 0}$ is implied by Lemma 9. In what follows, we show the asymptotic compactness of the solutions.

It follows from Lemma 4 that the precompactness in H^1 of a bounded sequence $u_n(t_n)$ is implied by the following two conditions:

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \limsup_{t \rightarrow \infty} \sup_n \int_{|x| > N} (|P_{\geq \delta} u_n(t)|^2 + |\nabla P_{\geq \delta} u_n(t)|^2) dx = 0, \quad (26)$$

$$\limsup_{N \rightarrow \infty} \limsup_{t \rightarrow \infty} \sup_n \int (|P_{> N} u_n(t)|^2 + |\nabla P_{> N} u_n(t)|^2) dx = 0. \quad (27)$$

In the sequel, we first prove (26) and then verify (27) when the forcing term g is small enough.

5.1. Uniform smallness on the complements of large balls

In this subsection, we verify the validity of (26) under the smallness condition on g . Consider the following energy functional $E_{\geq \delta}^N$: for $u \in H^1(\mathbf{R}^3)$:

$$E_{\geq \delta}^N(u) = \frac{1}{2} \int (u_{\geq \delta}^2 + |\nabla u_{\geq \delta}|^2) \chi(x/N) dx,$$

where χ is a smooth cut-off function such that $0 \leq \chi \leq 1$ and $\chi(x) = 0$ for $|x| \leq 1$; $\chi(x) = 1$ for $|x| > 2$. We now estimate $E_{\geq \delta}^N$ as follows. By the definition we have

$$\begin{aligned} \frac{d}{dt} E_{\geq \delta}^N(t) &= \int (u_{\geq \delta} \partial_t u_{\geq \delta} + \langle \nabla u_{\geq \delta}, \nabla \partial_t u_{\geq \delta} \rangle) \chi(x/N) dx \\ &= \nu \int u_{\geq \delta} (\Delta(1 - \Delta)^{-1} u_{\geq \delta}) \chi(x/N) dx \end{aligned} \quad (28)$$

$$+ \nu \int \langle \nabla u_{\geq \delta}, \Delta(1 - \Delta)^{-1} \nabla u_{\geq \delta} \rangle \chi(x/N) dx \quad (29)$$

$$- \int u_{\geq \delta} \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \chi(x/N) dx \quad (30)$$

$$- \int \langle \nabla u_{\geq \delta}, \nabla \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \rangle \chi(x/N) dx \quad (31)$$

$$+ \int u_{\geq \delta} (1 - \Delta)^{-1} g_{\geq \delta} \chi(x/N) dx \quad (32)$$

$$+ \int \langle \nabla u_{\geq \delta}, (1 - \Delta)^{-1} \nabla g_{\geq \delta} \rangle \chi(x/N) dx. \quad (33)$$

By the identity $\Delta(1 - \Delta)^{-1} = -1 + (1 - \Delta)^{-1}$, (29) can be rewritten as

$$\begin{aligned} & v \int \langle \nabla u_{\geq \delta}, \Delta(1 - \Delta)^{-1} \nabla u_{\geq \delta} \rangle \chi(x/N) dx \\ &= -v \int \langle \nabla u_{\geq \delta}, \nabla u_{\geq \delta} \rangle \chi(x/N) dx + v \int \langle \nabla u_{\geq \delta}, (1 - \Delta)^{-1} \nabla u_{\geq \delta} \rangle \chi(x/N) dx \\ &= -v \int |\nabla u_{\geq \delta}|^2 \chi(x/N) dx - v \int u_{\geq \delta} (\Delta(1 - \Delta)^{-1} u_{\geq \delta}) \chi(x/N) dx \\ &\quad - \frac{v}{N} \int u_{\geq \delta} \langle (1 - \Delta)^{-1} \nabla u_{\geq \delta}, \nabla \chi(x/N) \rangle dx, \end{aligned}$$

which implies

$$\begin{aligned} & v \int \langle \nabla u_{\geq \delta}, \Delta(1 - \Delta)^{-1} \nabla u_{\geq \delta} \rangle \chi(x/N) dx + v \int u_{\geq \delta} (\Delta(1 - \Delta)^{-1} u_{\geq \delta}) \chi(x/N) dx \\ &= -v \int |\nabla u_{\geq \delta}|^2 \chi(x/N) dx - \frac{v}{N} \int u_{\geq \delta} \langle (1 - \Delta)^{-1} \nabla u_{\geq \delta}, \nabla \chi(x/N) \rangle dx. \quad (34) \end{aligned}$$

Similarly, for (31) we have

$$\begin{aligned} & \int \langle \nabla u_{\geq \delta}, \nabla \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \rangle \chi(x/N) dx \\ &= - \int u_{\geq \delta} \left(\Delta(1 - \Delta)^{-1} \operatorname{div} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \right) \chi(x/N) dx \\ &\quad - \frac{1}{N} \int u_{\geq \delta} \langle \nabla \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta}, \nabla \chi(x/N) \rangle dx \\ &= \int u_{\geq \delta} \operatorname{div} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \chi(x/N) dx \\ &\quad - \int u_{\geq \delta} \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \chi(x/N) dx \\ &\quad - \frac{1}{N} \int u_{\geq \delta} \langle \nabla \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta}, \nabla \chi(x/N) \rangle dx, \end{aligned}$$

which shows

$$\begin{aligned} & \int \langle \nabla u_{\geq \delta}, \nabla \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \rangle \chi(x/N) dx \\ &+ \int u_{\geq \delta} \operatorname{div}(1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \chi(x/N) dx \end{aligned}$$

$$\begin{aligned}
&= \int u_{\geq \delta} \operatorname{div} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \chi(x/N) dx \\
&\quad - \frac{1}{N} \int u_{\geq \delta} \langle \nabla \operatorname{div} (1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta}, \nabla \chi(x/N) \rangle dx. \quad (35)
\end{aligned}$$

For the terms containing the forcing function, we obtain

$$\begin{aligned}
&\int u_{\geq \delta} (1 - \Delta)^{-1} g_{\geq \delta} \chi(x/N) dx + \int \langle \nabla u_{\geq \delta}, (1 - \Delta)^{-1} \nabla g_{\geq \delta} \rangle \chi(x/N) dx \\
&= \int u_{\geq \delta} g_{\geq \delta} \chi(x/N) dx - \frac{1}{N} \int u_{\geq \delta} \langle \nabla (1 - \Delta)^{-1} g_{\geq \delta}, \nabla \chi(x/N) \rangle dx. \quad (36)
\end{aligned}$$

It follows from (28)–(36) that

$$\begin{aligned}
&\frac{d}{dt} E_{\geq \delta}^N(t) + \nu \int |\nabla u_{\geq \delta}|^2 \chi(x/N) dx \\
&= -\frac{\nu}{N} \int u_{\geq \delta} \langle \nabla (1 - \Delta)^{-1} u_{\geq \delta}, \nabla \chi(x/N) \rangle dx \\
&\quad - \int u_{\geq \delta} \operatorname{div} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \chi(x/N) dx \\
&\quad + \frac{1}{N} \int u_{\geq \delta} \langle \nabla \operatorname{div} (1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta}, \nabla \chi(x/N) \rangle dx \\
&\quad + \int u_{\geq \delta} g_{\geq \delta} \chi(x/N) dx \\
&\quad - \frac{1}{N} \int u_{\geq \delta} \langle \nabla (1 - \Delta)^{-1} g_{\geq \delta}, \nabla \chi(x/N) \rangle dx. \quad (37)
\end{aligned}$$

By Corollary 2, we find

$$\int |\nabla u_{\geq \delta}|^2 \chi(x/N) dx \geq c \delta^2 \int |u_{\geq \delta}|^2 \chi(x/N) dx - \frac{C}{N^2 \delta^2} \|u\|_{H^1}^2.$$

Next, all terms containing the good term N^{-1} in front of them are easily estimated by Cauchy–Schwartz and Sobolev embeddings by

$$\frac{C_\nu}{N \delta^2} \left(\|u\|_{H^1}^2 + \|u\|_{H^1}^3 + \|g\|_{L^2} \|u\|_{H^1} \right).$$

Therefore, it follows from (37) that

$$\begin{aligned} \frac{d}{dt} E_{\geq \delta}^N(t) + cv\delta^2 E_{\geq \delta}^N &\leq - \int u_{\geq \delta} \operatorname{div} \left(u + \frac{1}{2} u^2 \right)_{\geq \delta} \chi(x/N) dx \\ &\quad + \int u_{\geq \delta} g_{\geq \delta} \chi(x/N) dx \\ &\quad + \frac{C_v}{N\delta^2} \left(\|u\|_{H^1}^2 + \|u\|_{H^1}^3 + \|g\|_{L^2} \|u\|_{H^1} \right). \end{aligned} \quad (38)$$

For the forcing term, we have

$$\begin{aligned} &\left| \int u_{\geq \delta} g_{\geq \delta} \chi(x/N) dx \right| \\ &\leq \sqrt{E_{\geq \delta}^N} \int g^2(x) \chi(x/N) dx \\ &\leq \frac{cv\delta^2}{2} E_{\geq \delta}^N + \frac{1}{2cv\delta^2} \|g_{\geq \delta}\|_{L^2(|x| \geq N)}^2. \end{aligned}$$

The estimate on $\int u_{\geq \delta} \operatorname{div}(u^2)_{\geq \delta} \chi(x/N) dx$ is a little more intricate. We write

$$\begin{aligned} &\int u_{\geq \delta} \operatorname{div}(u^2)_{\geq \delta} \chi(x/N) dx \\ &= \int (u - u_{< \delta}) \operatorname{div}(u^2 - (u^2)_{< \delta}) \chi(x/N) dx \\ &= -\frac{1}{3N} \int u^3 \operatorname{div} \chi(x/N) dx - \int u \operatorname{div}((u^2)_{< \delta}) \chi(x/N) dx \\ &\quad - \int u_{< \delta} \operatorname{div}(u^2) \chi(x/N) dx + \int u_{< \delta} \operatorname{div}(u^2)_{< \delta} \chi(x/N) dx. \end{aligned}$$

The first term above is estimated by Sobolev embedding by $\frac{C}{N} \|u\|_{H^1}^3$. For the second term, we have by Cauchy–Schwartz and the Bernstein inequality:

$$\left| \int u \operatorname{div}((u^2)_{< \delta}) \chi(x/N) dx \right| \lesssim \|u\|_{L^2} \left\| \operatorname{div}((u^2)_{< \delta}) \right\|_{L^2} \lesssim \delta^{n/2+1} \|u\|_{L^2}^3.$$

Clearly, the fourth term can be estimated analogously. For the third term, integration by parts allows us to estimate

$$\begin{aligned} \left| \int u_{< \delta} \operatorname{div}(u^2) \chi(x/N) dx \right| &\lesssim \|\nabla u_{< \delta}\|_{L^\infty} \|u^2\|_{L^1} + N^{-1} \|u_{< \delta}\|_{L^\infty} \|u^2\|_{L^1} \\ &\lesssim \delta^{n/2+1} \|u\|_{L^2}^3 + \delta^{n/2} N^{-1} \|u\|_{L^2}^3. \end{aligned}$$

Plugging the above estimates into (38) we find

$$\begin{aligned} \frac{d}{dt} E_{\geq \delta}^N(t) + \frac{c\nu\delta^2}{2} E_{\geq \delta}^N &\leq \frac{C_\nu}{N\delta^2} \left(\|u\|_{H^1}^2 + \|u\|_{H^1}^3 + \|g\|_{L^2}^2 \right) + \delta^{n/2+1} \|u\|_{L^2}^3 \\ &\quad + \frac{C_\nu}{\delta^2} \|g_{\geq \delta}\|_{L^2(|x|>N)}^2. \end{aligned}$$

Then Lemma 9 and Gronwall lemma imply (26).

5.2. Uniform continuity in H^1

We now verify (27) under the smallness condition on g . Notice that for large N :

$$\int |\nabla u_{>N}|^2 dx \gg \int |u_{>N}|^2 dx.$$

Therefore, when N is sufficiently large, (27) is equivalent to the following:

$$J_{>N}(t) = \frac{1}{2} \int |\nabla u_{>N}(t)|^2 dx \rightarrow 0 \quad \text{uniformly in time as } N \rightarrow \infty. \quad (39)$$

In what follows, we check the validity of (39). Before doing that, we need the following inequality which is concerned with the cut-off of the nonlinear term in the equation.

Lemma 10. *For every positive integer l , we have*

$$\left\| (u^2)_{>2^l} \right\|_{L^2(\mathbf{R}^3)} \leq C 2^{-l/2} \|u\|_{H^1}^2,$$

where C is a constant independent of l .

Proof. For simplicity, we will write $P_{2^l}u$ as u_l etc., in the sequel. Note that

$$P_{2^l}(u^2) = \sum_{k,m} P_{2^l}(u_k u_m), \quad (40)$$

where all the terms satisfying $\max(k, m) \leq l-3$ are zero, and so are the terms satisfying both $\min(k, m) \geq l+2$ and $|k-m| \geq 2$. Thus, we only need to consider the following two types of terms:

$$\text{high-low interaction:} \quad \sum_{k,m: |k-l| \leq 3, m \leq k} P_{2^l}(u_k u_m),$$

$$\text{high-high interaction:} \quad \sum_{k,m: |k-m| \leq 2, \min(k,m) \geq l+2} P_{2^l}(u_k u_m).$$

For the high–low interactions, by the Bernstein inequality, we have

$$\begin{aligned}
 & \left\| \sum_{k,m: |k-l| \leq 3, m \leq k} P_{2^l}(u_k u_m) \right\|_{L^2} \\
 & \lesssim \|u_{l-3 < \cdot < l+3}\|_{L^2} \sum_{m \leq l+3} \|u_m\|_{L^\infty} \\
 & \lesssim \|\nabla u_{l-3 < \cdot < l+3}\|_{L^2} 2^{-l} \sum_{m \leq l+3} 2^{3m/2} \|u_m\|_{L^2} \\
 & \lesssim \|\nabla u\|_{L^2} 2^{-l} \left(\sum_{m \leq l+3} 2^m \right)^{1/2} \left(\sum_{m \leq l+3} 2^{2m} \|u_m\|_{L^2}^2 \right)^{1/2} \lesssim 2^{-l/2} \|\nabla u\|_{L^2}^2. \quad (41)
 \end{aligned}$$

For the high–high interaction, we get by the Bernstein inequality

$$\begin{aligned}
 & \left\| \sum_{k,m: |k-m| \leq 2, \min(k,m) \geq l+2} P_{2^l}(u_k u_m) \right\|_{L^2} \\
 & \lesssim 2^{3l/2} \sum_{k,m: |k-m| \leq 2, \min(k,m) \geq l+2} \|P_{2^l}(u_k u_m)\|_{L^1} \\
 & \lesssim 2^{3l/2} \sum_{k,m: |k-m| \leq 2, \min(k,m) \geq l+2} \|u_k\|_{L^2} \|u_m\|_{L^2} \\
 & \lesssim 2^{3l/2} \sum_{k \geq l+2} 2^{-2k} \|\nabla u_k\|_{L^2} \|\nabla u_{k-2 \leq \cdot \leq k+2}\|_{L^2} \lesssim 2^{-l/2} \|u\|_{H^1}^2. \quad (42)
 \end{aligned}$$

Then it follows from (40)–(42) that

$$\left\| P_{2^l}(u^2) \right\|_{L^2(\mathbb{R}^3)} \lesssim 2^{-l/2} \|u\|_{H^1}^2,$$

which implies Lemma 10. $\square$

We are now in a position to prove (39) by using Lemma 10. By Eq. (1) we find

$$\begin{aligned}
 \partial_t J_{>N} &= \int \langle \nabla u_{>N}, \nabla \partial_t u_{>N} \rangle dx \\
 &= v \int \langle \nabla u_{>N}, \nabla \Delta(1 - \Delta)^{-1} u_{>N} \rangle dx
 \end{aligned}$$

$$\begin{aligned}
& - \int \left\langle \nabla u_{>N} \operatorname{div} \nabla (1 - \Delta)^{-1} \left(u + \frac{1}{2} u^2 \right)_{>N} \right\rangle dx \\
& + \int \langle \nabla u_{>N}, (1 - \Delta)^{-1} g_{>N} \rangle dx.
\end{aligned} \quad (43)$$

Next, we estimate each term on the right-hand side of (43). The first term on the right-hand side of (43) is bounded by

$$v \int \langle \nabla u_{>N}, \nabla \Delta (1 - \Delta)^{-1} u_{>N} \rangle dx = -v \left\| \sqrt{-\Delta (1 - \Delta)^{-1}} \nabla u_{>N} \right\|_{L^2}^2 \lesssim -v \|\nabla u_{>N}\|_{L^2}^2. \quad (44)$$

By Lemma 10 and $\operatorname{div} \nabla (1 - \Delta)^{-1} : L^2 \rightarrow L^2$, the second term on the right-hand side of (43) is bounded by

$$\|\nabla u_{>N}\|_{L^2} \left\| \operatorname{div} \nabla (1 - \Delta)^{-1} (u^2)_{>N} \right\|_{L^2} \leq C \|u\|_{H^1} \left\| (u^2)_{>N} \right\|_{L^2} \leq C N^{-1/2} \|u\|_{H^1}^3. \quad (45)$$

For the forcing g , we have

$$\left| \int \langle \nabla u_{>N}, (1 - \Delta)^{-1} g_{>N} \rangle dx \right| \leq \|\nabla u_{>N}\|_{L^2} \|(1 - \Delta)^{-1} g_{>N}\|_{L^2} \leq N^{-2} \|u\|_{H^1} \|g\|_{L^2}. \quad (46)$$

It follows from (43)–(46) and Lemma 9 that, for $N \geq 1$:

$$\frac{d}{dt} J_{>N}(t) + 2v J_{>N}(t) \leq C N^{-1/2}.$$

Then, by Gronwall's lemma, we get, for all $t \geq 0$:

$$J_{>N}(t) \leq e^{-2vt} J_{>N}(0) + C N^{-1/2},$$

which implies (39). The proof is complete.

Next, we establish the asymptotic compactness of the solutions which follows immediately from (26) and (27).

Lemma 11. *Given $R > 0$. Let $\varepsilon = \varepsilon(R, v, \mu)$ be the constant in Lemma 9. Assume that $t_n \rightarrow \infty$, $\|u_n\|_{H^1} \leq R$ and $\|g\|_{L^2} + G \leq \varepsilon$. Then $\{S(t_n)u_n\}$ has a convergent subsequence in $H^1(\mathbf{R}^3)$.*

Proof. We check the conditions of Lemma 4 for the sequence $\{S(t_n)u_n\}$. By Lemma 9 we know $\{S(t_n)u_n\}$ is bounded in $H^1(\mathbf{R}^3)$, and by (26) and (27) we find $\{S(t_n)u_n\}$ satisfies (14) and (15). Therefore, it follows from Lemma 4 that $\{S(t_n)u_n\}$ is precompact. $\square$

We are now ready to show one of our main results, i.e., the existence of attractors for the dynamical system $S(t)_{t \geq 0}$.

Theorem 3. *Given a ball $B = \{u \in H^1(\mathbf{R}^3) : \|u\|_{H^1} \leq R\}$. Let $\varepsilon = \varepsilon(R, v, \mu)$ be the constant in Lemma 9. If $\|g\|_{L^2} + G \leq \varepsilon$, then $S(t)_{t \geq 0}$ has an attractor $\mathcal{A}$ in $H^1(\mathbf{R}^3)$ which is a compact invariant set and attracts every bounded subset of B with respect to the norm topology of $H^1(\mathbf{R}^3)$.*

Proof. By Lemma 9 we see $S(t)_{t \geq 0}$ is point dissipative in $H^1(\mathbf{R}^3)$, and by Lemma 11 we know $S(t)_{t \geq 0}$ is asymptotic compact. Therefore it follows from a standard result (see, e.g., [9,10,29,30,45]) that $S(t)_{t \geq 0}$ has an attractor in B . The proof is complete. $\square$

6. Regularity of attractors

In this section, we establish the regularity of attractors. More precisely, we prove that attractor $\mathcal{A}$ in $H^1(\mathbf{R}^3)$ is actually a bounded subset of $H^2(\mathbf{R}^3)$. For that purpose, we decompose the high-frequency of the solution into two parts: one part is bounded in $H^2(\mathbf{R}^3)$ and the other is asymptotically null as $t \rightarrow \infty$.

In what follows, we denote by χ the characteristic function of the unit ball in $\mathbf{R}^3$. Given a number $N > 0$, let P_N be the projection operator from $L^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}^3)$ defined by

$$\widehat{P_N u}(\xi) = \chi\left(\frac{\xi}{N}\right) \hat{u}(\xi).$$

For convenience, we also denote by $Q_N = I - P_N$. Note that the projection operator defined above is different from the Littlewood–Paley projection operator given in Section 2.

We now decompose the solution into its low-frequency and high-frequency parts. Given $u_0 \in H^1(\mathbf{R}^3)$, let $u(t) = S(t)u_0$, $p(t) = P_N u(t)$ and $q(t) = Q_N u(t)$. It is clear $u(t) = p(t) + q(t)$. We further split the high-frequency q as $q(t) = y(t) + z(t)$ where y is the solution of the following initial value problem:

$$y_t - \Delta y_t - v \Delta y + \operatorname{div}(y) + Q_N((p + y) \operatorname{div}(p + y)) = Q_N g, \quad (47)$$

$$y(0) = 0. \quad (48)$$

The following lemma shows that problem (47)–(48) is well-posed.

Lemma 12. *Assume $g \in L^2(\mathbf{R}^3)$ and $u_0 \in H^1(\mathbf{R}^3)$. Then for every $N > 0$, there exists a unique solution y of problem (47)–(48) such that $y \in C([0, \infty), Q_N H^1(\mathbf{R}^3))$.*

Proof. The proof of this lemma is similar to that of well-posedness of the BBM equation (1), and therefore the details are omitted here. $\square$

In the following, we derive estimates on y when $u_0 \in \mathcal{A}$, the attractor, and show that $y(t)$ ($t \geq 0$) is bounded in $H^2(\mathbf{R}^3)$. If $u_0 \in \mathcal{A}$, then by the invariance of $\mathcal{A}$, we have $u(t) = S(t)u_0 \in \mathcal{A}$ for all $t \geq 0$. Therefore, there exists a constant C depending on v and g (but independent of N) such that, for all $t \geq 0$,

$$\|u(t)\|_{H^1} \leq C, \quad (49)$$

and

$$\|q(t)\|_{H^1} \leq C, \quad \|p(t)\|_{H^1} \leq C, \quad \|p(t)\|_{H^2} \leq CN. \quad (50)$$

We now derive the estimates for y in $H^1(\mathbf{R}^3)$.

Lemma 13. Assume $g \in L^2(\mathbf{R}^3)$ and $u_0 \in \mathcal{A}$. Then there exists N_0 depending on v and g such that for all $N \geq N_0$ and $t \geq 0$, the solution y of problem (47)–(48) satisfies

$$\|y(t)\|_{H^1} \leq C,$$

where C depends only on v and g .

Proof. Multiplying (47) by y and then integrating, we get

$$\frac{1}{2} \frac{d}{dt} \left(\|y\|^2 + \|\nabla y\|^2 \right) + v \|\nabla y\|^2 = \int_{\mathbf{R}^3} g y \, dx - \int_{\mathbf{R}^3} y(p + y) \operatorname{div}(p + y) \, dx. \quad (51)$$

Next, we estimate each term on the right-hand side of (51). The first term on the right-hand side of (51) is bounded by

$$\left| \int_{\mathbf{R}^3} g y \, dx \right| \leq \|g\| \|y\| \leq C_1 N^{-1} \|\nabla y\| \leq C + N^{-2} \|\nabla y\|^2. \quad (52)$$

Notice that the second term on the right-hand side of (51) can be rewritten as

$$\begin{aligned} & \int_{\mathbf{R}^3} y(p + y) \operatorname{div}(p + y) \, dx \\ &= \int_{\mathbf{R}^3} p y \operatorname{div}(p) \, dx + \int_{\mathbf{R}^3} y^2 \operatorname{div}(y) \, dx + \int_{\mathbf{R}^3} y \operatorname{div}(p y) \, dx. \end{aligned} \quad (53)$$

For the first term on the right-hand side of (53), by (50) we have

$$\begin{aligned} \left| \int_{\mathbf{R}^3} py \operatorname{div}(p) dx \right| &= \left| \frac{1}{2} \int_{\mathbf{R}^3} p^2 \operatorname{div}(y) dx \right| \leq \frac{1}{2} \|p\|_4^2 \|\nabla y\| \leq C_1 \|p\|_{H^1}^2 \|\nabla y\| \\ &\leq C_2 \|\nabla y\| \leq C + \frac{1}{4} v \|\nabla y\|^2. \end{aligned} \quad (54)$$

The second term on the right-hand side of (53) is actually zero:

$$\int_{\mathbf{R}^3} y^2 \operatorname{div}(y) dx = \frac{1}{3} \int_{\mathbf{R}^3} \operatorname{div}(y^3) dx = 0. \quad (55)$$

By (50), the last term on the right-hand side of (53) is bounded by

$$\begin{aligned} \left| \int_{\mathbf{R}^3} y \operatorname{div}(py) dx \right| &= \left| \int_{\mathbf{R}^3} py \operatorname{div}(y) dx \right| \leq C_1 \|p\|_6 \|y\|_3 \|\nabla y\| \\ &\leq C_2 \|p\|_{H^1} \|y\|^{\frac{1}{2}} \|\nabla y\|^{\frac{3}{2}} \leq CN^{-\frac{1}{2}} \|\nabla y\|^2. \end{aligned} \quad (56)$$

It follows from (53)–(56) that

$$\int_{\mathbf{R}^3} y(p+y) \operatorname{div}(p+y) dx \leq C + CN^{-\frac{1}{2}} \|\nabla y\|^2 + \frac{1}{4} v \|\nabla y\|^2. \quad (57)$$

From (51), (52) and (57) we find

$$\frac{1}{2} \frac{d}{dt} (\|y\|^2 + \|\nabla y\|^2) + v \|\nabla y\|^2 \leq C + (CN^{-\frac{1}{2}} + N^{-2}) \|\nabla y\|^2 + \frac{1}{4} v \|\nabla y\|^2.$$

Let $N_0 = \max\{1, \frac{16(C+1)^2}{v^2}\}$. Then for all $N \geq N_0$ we get

$$\frac{d}{dt} (\|y\|^2 + \|\nabla y\|^2) + v \|\nabla y\|^2 \leq C. \quad (58)$$

Since $y \in Q_N H^1(\mathbf{R}^3)$, the following inequality holds, for $N \geq N_0$,

$$v \|\nabla y\|^2 \geq \frac{1}{2} v \|\nabla y\|^2 + \frac{1}{2} v N \|y\|^2 \geq \frac{1}{2} v (\|\nabla y\|^2 + \|y\|^2). \quad (59)$$

By (58)–(59) we obtain, for all $t \geq 0$ and $N \geq N_0$,

$$\frac{d}{dt} (\|y\|^2 + \|\nabla y\|^2) + \frac{1}{2} v (\|\nabla y\|^2 + \|y\|^2) \leq C.$$

Then Lemma 13 follows from (48) and Gronwall's lemma.

Lemma 14. Assume $g \in L^2(\mathbf{R}^3)$ and $u_0 \in \mathcal{A}$. Then there exists N_0 depending on v and g such that for all $N \geq N_0$ and $t \geq 0$, the solution y of problem (47)–(48) satisfies

$$\|y(t)\|_{H^2}^2 \leq C(1 + N),$$

where C depends only on v and g , but not on N .

Proof. Multiplying (47) by $-\Delta y$ and then integrating, we find

$$\frac{1}{2} \frac{d}{dt} (\|\nabla y\|^2 + \|\Delta y\|^2) + v \|\Delta y\|^2 = - \int_{\mathbf{R}^3} g \Delta y \, dx + \int_{\mathbf{R}^3} (p+y) \operatorname{div}(p+y) \Delta y \, dx. \quad (60)$$

The first term on the right-hand side of (60) is bounded by

$$\left| \int_{\mathbf{R}^3} g \Delta y \, dx \right| \leq \|g\| \|\Delta y\| \leq C + \frac{1}{8} v \|\Delta y\|^2. \quad (61)$$

For the second term on the right-hand side of (61), we have the following identity:

$$\begin{aligned} \int_{\mathbf{R}^3} (p+y) \operatorname{div}(p+y) \Delta y \, dx &= \int_{\mathbf{R}^3} p \operatorname{div}(p) \Delta y \, dx + \int_{\mathbf{R}^3} y \operatorname{div}(y) \Delta y \, dx \\ &\quad + \int_{\mathbf{R}^3} \operatorname{div}(py) \Delta y \, dx. \end{aligned} \quad (62)$$

Next, we estimate each term on the right-hand side of (62). By (50), we find the following bound for the first term on the right-hand side of (62):

$$\begin{aligned} \left| \int_{\mathbf{R}^3} p \operatorname{div}(p) \Delta y \, dx \right| &\leq \|p\|_6 \|\operatorname{div}(p)\|_3 \|\Delta y\| \leq C_1 \|p\|_{H^1}^{\frac{3}{2}} \|p\|_{H^2}^{\frac{1}{2}} \|\Delta y\| \\ &\leq C_1 N^{\frac{1}{2}} \|p\|_{H^1}^2 \|\Delta y\| \leq C_2 N^{\frac{1}{2}} \|\Delta y\| \\ &\leq CN + \frac{1}{8} v \|\Delta y\|^2. \end{aligned} \quad (63)$$

By Lemma 13, we see the second term on the right-hand side of (62) satisfies

$$\begin{aligned} \left| \int_{\mathbf{R}^3} y \operatorname{div}(y) \Delta y \, dx \right| &\leq \|y\|_6 \|\operatorname{div}(y)\|_3 \|\Delta y\| \leq C_1 \|y\|_{H^1}^{\frac{3}{2}} \|\Delta y\|^{\frac{3}{2}} \\ &\leq C_2 \|\Delta y\|^{\frac{3}{2}} \leq C + \frac{1}{8} v \|\Delta y\|^2, \end{aligned} \quad (64)$$

for $N \geq N_0$ where N_0 is the constant in Lemma 13. Similarly, we also have

$$\left| \int_{\mathbf{R}^3} \operatorname{div}(py) \Delta y \, dx \right| \leq C + CN + \frac{1}{8} v \|\Delta y\|^2. \quad (65)$$

By (62)–(65) we see there exists a constant C , depending on v and g , such that

$$\left| \int_{\mathbf{R}^3} (p+y) \operatorname{div}(p+y) \Delta y \, dx \right| \leq C + CN + \frac{3}{8} v \|\Delta y\|^2. \quad (66)$$

It follows from (60), (61) and (66) that, for all $t \geq 0$,

$$\frac{d}{dt} (\|\nabla y\|^2 + \|\Delta y\|^2) + v \|\Delta y\|^2 \leq C + CN. \quad (67)$$

Since

$$v \|\Delta y\|^2 \geq \frac{1}{2} v \|\Delta y\|^2 + \frac{1}{2} v C_1 N^2 \|\nabla y\|^2 \geq \frac{1}{2} v (\|\nabla y\|^2 + \|\Delta y\|^2), \quad (68)$$

for $N \geq N_1 = C_1^{-1/2}$. Let $N_2 = \max\{N_0, N_1\}$. Then from (67)–(68) we get, for all $N \geq N_2$ and $t \geq 0$,

$$\frac{d}{dt} (\|\nabla y\|^2 + \|\Delta y\|^2) + \frac{1}{2} v (\|\nabla y\|^2 + \|\Delta y\|^2) \leq C + CN,$$

which and Gronwall's lemma imply Lemma 14. $\square$

Projecting Eq. (47) onto $\mathcal{Q}_N H^1(\mathbf{R}^3)$, we find that $q = \mathcal{Q}_N u$ satisfies

$$q_t - \Delta q_t - v \Delta q + \operatorname{div}(q) + \mathcal{Q}_N ((p+q) \operatorname{div}(p+q)) = \mathcal{Q}_N g, \quad (69)$$

$$q(0) = \mathcal{Q}_N u_0. \quad (70)$$

Then from (69) and (47), it follows that $z = q - y$ satisfies

$$z_t - \Delta z_t - v \Delta z + \operatorname{div}(z) = -\mathcal{Q}_N (z \operatorname{div}(q) + y \operatorname{div}(z) + \operatorname{div}(pz)), \quad (71)$$

$$z(0) = \mathcal{Q}_N u_0. \quad (72)$$

In the sequel, we show z converges to zero as t goes to infinity, more precisely, we have the following lemma.

Lemma 15. Assume $g \in L^2(\mathbf{R}^3)$ and $u_0 \in \mathcal{A}$. Then there exists N_0 depending on v and g such that for all $N \geq N_0$ and $t \geq 0$, the solution z of problem (71)–(72)

satisfies

$$\|z(t)\|_{H^1}^2 \leq C e^{-\frac{1}{2}vt},$$

where C depends only on v and g .

Proof. Multiplying (71) by z and then integrating, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|z\|^2 + \|\nabla z\|^2) + v \|\nabla z\|^2 = & - \int_{\mathbf{R}^3} z^2 \operatorname{div}(q) \, dx - \int_{\mathbf{R}^3} yz \operatorname{div}(z) \, dx \\ & - \int_{\mathbf{R}^3} z \operatorname{div}(pz) \, dx. \end{aligned} \quad (73)$$

For the first term on the right-hand side of (73), by (50), we have the following estimates:

$$\left| \int_{\mathbf{R}^3} z^2 \operatorname{div}(q) \, dx \right| \leq \|z\|_4^2 \|\operatorname{div}(q)\| \leq C_1 \|q\|_{H^1} \|z\|_{H^{3/4}}^2 \leq C_2 N^{-\frac{1}{2}} \|\nabla z\|^2. \quad (74)$$

Similarly, the last two terms on the right-hand side of (73) are also bounded by $C_2 N^{-\frac{1}{2}} \|\nabla z\|^2$. Then it follows from (73)–(74) that

$$\frac{1}{2} \frac{d}{dt} (\|z\|^2 + \|\nabla z\|^2) + v \|\nabla z\|^2 \leq C_3 N^{-\frac{1}{2}} \|\nabla z\|^2. \quad (75)$$

Let $N \geq N_1 = \frac{4C_3^2}{v^2}$. Then we find, for $N \geq N_1$ and $t \geq 0$,

$$\frac{d}{dt} (\|z\|^2 + \|\nabla z\|^2) + v \|\nabla z\|^2 \leq 0.$$

Again, by

$$\|\nabla z\|^2 \geq \frac{1}{2} \|\nabla z\|^2 + C_4 N^2 \|z\|^2 \geq \frac{1}{2} (\|z\|^2 + \|\nabla z\|^2), \quad (76)$$

for $N \geq N_2 = (2C_4)^{-\frac{1}{2}}$. Let $N_3 = \max\{N_1, N_2\}$. By (75)–(76) we get, for all $N \geq N_3$ and $t \geq 0$,

$$\frac{d}{dt} (\|z\|^2 + \|\nabla z\|^2) + \frac{1}{2} v (\|z\|^2 + \|\nabla z\|^2) \leq 0.$$

Then Lemma 15 follows. $\square$

Next, we show that the attractor is bounded in $H^2(\mathbf{R}^3)$. For that purpose, we split the semigroup $S(t)_{t \geq 0}$ as $S(t) = S_1(t) + S_2(t)$. Given $u_0 \in H^1(\mathbf{R}^3)$, for each $t \geq 0$, we define

$$S_1(t)u_0 = p(t) + y(t), \quad S_2(t)u_0 = z(t),$$

where $p(t) = P_N u(t) = P_N S(t)u_0$, $y(t)$ is the solution of problem (47)–(48), and $z(t)$ is the solution of problem (71)–(72). It is evident that

$$S(t) = S_1(t) + S_2(t), \quad t \geq 0. \quad (77)$$

We are now ready to prove the regularity of the attractor.

Proof of Theorem 2. Assume $u \in \mathcal{A}$ and $t_n \rightarrow \infty$. Then by the invariance of $\mathcal{A}$, there exists a sequence $\{u_n\} \subseteq \mathcal{A}$ such that

$$u = S(t_n)u_n \quad \text{for all } n \geq 1.$$

By (77) we find

$$u = S_1(t_n)u_n + S_2(t_n)u_n \quad \text{for all } n \geq 1. \quad (78)$$

From Lemma 14 we see there exist C and N_0 depending only on v and g such that

$$\|S_1(t_n)u_n\|_{H^2} \leq C + CN_0^{\frac{1}{2}}. \quad (79)$$

Therefore, there exist subsequences of u_n and t_n (still denoted by u_n and t_n) and $v \in H^2(\mathbf{R}^3)$ such that

$$S_1(t_n)u_n \rightarrow v \quad \text{weakly in } H^2(\mathbf{R}^3). \quad (80)$$

Moreover, we have

$$\|v\|_{H^2} \leq \liminf_{n \rightarrow \infty} \|S_1(t_n)u_n\|_{H^2} \leq C \left(1 + N_0^{\frac{1}{2}}\right). \quad (81)$$

Consider now $\phi \in H^1(\mathbf{R}^3)$. From (78), we have

$$(u, \phi) = (S_1(t_n)u_n, \phi) + (S_2(t_n)u_n, \phi),$$

where $(\cdot, \cdot)$ denotes the inner product of $H^1(\mathbf{R}^3)$. Taking the limit as $n \rightarrow \infty$, by (80) and Lemma 15, we get $(u, \phi) = (v, \phi)$ for every $\phi \in H^1(\mathbf{R}^3)$, which implies $u = v$. Then, it follows from (81) that $u = v \in H^2(\mathbf{R}^3)$ and

$$\|u\|_{H^2} \leq C \left(1 + N_0^{\frac{1}{2}}\right).$$

The proof is complete. $\square$

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